(1a) Show that if \( \vec{p} \) is a constant vector, then \( \phi \equiv \vec{p} \cdot \vec{\nabla} (1/r) \) satisfies Laplace’s equation for \( r > 0 \).

(1b) Obtain the expression for the potential \( \phi \) in part (1a) as a fully explicit expansion in spherical harmonics. (i.e. a special case of the general expansion given in eqn (4.114) of the lectures.) [Note: The answer involves just a finite (and small) number of terms. The coefficients are completely explicit.]

(2a) Consider the angular momentum generators

\[
L_1 = -i(y \partial_z - z \partial_y), \\
L_2 = -i(z \partial_x - x \partial_z), \\
L_3 = -i(x \partial_y - y \partial_x),
\]

(where \( \partial_x \equiv \partial/\partial x \), etc.) Using the relations \( x = r \sin \theta \cos \varphi \), \( y = r \sin \theta \sin \varphi \), \( z = r \cos \theta \), show that

\[
L_+ = e^{i\varphi} \left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right), \\
L_3 = -i \frac{\partial}{\partial \varphi},
\]

where \( L_\pm \equiv L_1 \pm i L_2 \).

(2b) Using the generalised Rodrigues formula (eqn (4.90) in the lectures) and the definition (4.101) of the spherical harmonics, derive expressions for \( L_3 Y_{\ell m} \) and \( L_\pm Y_{\ell m} \). (In each case, the answer is a constant factor times a single spherical harmonic.)

(3) Many properties of the Bessel functions can be derived from a generating function. This defines the \( J_n(x) \) (for \( n \) an integer) as follows:

\[
G(x, t) \equiv e^\frac{1}{2}x(t^{-1}) = \sum_{n=-\infty}^{\infty} t^n J_n(x)
\]

All the following problems should be solved using this definition.

(3a) Show that \( J_n(x) \) as defined above does indeed satisfy Bessel’s equation \( x^2 J''_n + xJ'_n + (x^2 - n^2)J_n = 0 \). (The proof is analogous to the one given in the lectures for the Legendre polynomials.)

(3b) Show that \( J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x) \).

(3c) Show that \( J_{n-1}(x) + J_{n+1}(x) = (2n/x)J_n(x) \).

(3d) Show that \( J_{-n}(x) = (-1)^n J_n(x) \).

Due on Tuesday 19th March