Problem Sheet 1

(1a) Plug the expressions for the electric and magnetic fields in terms of gauge potentials (eqn (1.37) in the notes) into the Maxwell field equations in free space (in eqn (1.3)), and hence obtain the equations satisfied by $\phi$ and $\vec{A}$.

(1b) Obtain the simpler equations that result if $\phi$ and $\vec{A}$ satisfy the Lorenz Gauge condition $\nabla \cdot \vec{A} + (1/c) \partial \phi / \partial t = 0$.

(1c) Perform a gauge transformation (eqn (1.41) in the notes) on $\phi$ and $\vec{A}$ that satisfy the Lorenz gauge condition. Derive the equation that the gauge parameter $\lambda$ must satisfy if the transformed gauge potentials are required also to satisfy the Lorenz gauge condition.

(2) Show by explicit direct substitution, using eqn (1.71) in the notes, that if

$$\phi(\vec{r}) = \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d^3r',$$

then $\phi$ satisfies $\nabla^2 \phi(\vec{r}) = -4\pi \rho(\vec{r})$.

(3) By applying Green’s theorem (eqn (1.81) in the notes) with $\phi(\vec{r}) = G(\vec{r}_1, \vec{r})$ and $\psi(\vec{r}) = G(\vec{r}_2, \vec{r})$, prove that a Dirichlet Green function is necessarily symmetric in its arguments; $G_D(\vec{r}_1, \vec{r}_2) = G_D(\vec{r}_2, \vec{r}_1)$.

(4a) A spherically-symmetric charge distribution gives rise to the potential

$$\phi = \frac{q e^{-\mu r}}{r} \left(1 + \frac{\mu r}{2}\right)$$

everywhere, where $q$ and $\mu$ are constants. Find the charge distribution $\rho(r)$ that gives rise to this potential. (Be sure to consider $r = 0$ carefully, as well as $r > 0$. i.e., consider delta-function terms also.)

(4b) Integrate your results for the bulk charge distribution (i.e. the expression for $\rho(\vec{r})$ with $r > 0$), and for the delta-function contribution, over the volume of all space, and hence obtain the total electric charge. Discuss why this result is reasonable. (Consider a Gauss’ law integral of $\int \vec{E} \cdot d\vec{S}$ at infinity.)

Due on Thursday 2nd February