Celestial Mechanics and Orbital Motions

• Kepler’s Laws
• Newton’s Laws
• Tidal Forces
Tycho Brahe (1546-1601)

Foremost astronomer after the death of Copernicus. King Frederick II of Denmark set him up at Uraniborg, an observatory on the island of Hveen.

With new instruments (quadrant), Brahe could measure positions of stars and planets to 4 minutes of arc (1/8th the diameter of the full moon).

With new instruments (quadrant, similar to present-day sextant), Brahe could measure positions of stars and planets to better than 4 minutes of arc (1/8th the diameter of the full moon). Made the most accurate observations of planet positions ever recorded.
Using the Quadrant (similar to a Sextant) to measure a star’s altitude. An observer sights a star along the Quadrant while a plumb line measures the angle.
Tycho Brahe (1546-1601)

Also credited with observing the supernova of 1572, demonstrating that the stars do change. He coined the term “Nova” for “new star” in a book. Term still used today.

Brahe failed to find evidence for the Earth’s motion and presumed the Copernican model was false. Brahe endorsed the Geocentric cosmology.
Johannes Kepler (1571-1630)

Brahe moved to Prague in 1599. Kepler joined him to develop cosmological model consistent with Brahe’s observations.

Kepler initially obtained excellent agreement with Brahe’s data with the planets moving in spheres and equants, similar to Ptolemy’s model, except for 2 data points which were off by 8 minutes of arc (twice the accuracy of Brahe’s measurements).

Kepler kept struggling and finally rejected Ptolemaic model and eventually realized that that Brahe’s data were only consistent with a model where planets (1) orbit the Sun and (2) their orbits are ellipses.
Observed Planet and uncertainty on the measurement
Observed Planet and uncertainty on the measurement

Position Predicted by Model
Kepler’s Laws

Johannes Kepler (1571-1630)


1st Law: A planet orbits the Sun in an ellipse, with the Sun at one focus of the ellipse.

2nd Law: A line connecting a planet to the Sun sweeps out equal areas in equal time intervals.

3rd Law: The square of the orbital period, $P$, of a planet is equal to the cube of the average distance, $a$, of the planet from the Sun.

$$P^2 = kr^3$$
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Kepler’s First Law

\[ r' + r = 2a \]

defines an ellipse

Perihelion distance = \( a (1-e) \)
Kepler’s First Law

Perihelion distance = $a (1-e)$

$r' + r = 2a$
defines an ellipse

Perihelion distance = $a (1-e)$

“Perilously close”
Eccentricity

c Eccentricity describes how much an ellipse deviates from a perfect circle.
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Eccentricity = 1
Parabolic orbit (goes through perihelion only once)
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Kepler’s Second Law

Equal Areas = Equal Times
Conservation of Angular Momentum

Near perihelion, in any particular amount of time (such as 30 days) a planet sweeps out an area that is short by wide.

Near aphelion, in the same amount of time (such as 30 days) a planet sweeps out an area that is long by narrow.

The areas swept out during any 30-day period are all equal.
Kepler’s Second Law

Equal Areas = Equal Times
Conservation of Angular Momentum

The areas swept out during any 30-day period are all equal
Kepler’s Third Law

![Graph showing the relationship between average orbital speed (km/s) and average distance from the Sun (AU).]

This graph shows how orbital speed depends on distance from the Sun. (Kepler knew the form of this relationship but not the actual speeds, because the numerical value of the astronomical unit was not yet known.)
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Kepler's Third Law

\[ P^2 = kr^3 \]
What is the Physical Origin of Kepler’s Laws?

Combination of Gravitational Force and Conservation of Angular Momentum.
Newton’s Laws

Sir Isaac Newton (1643-1727)

Developed Universal theory of Gravity, and set his three Laws of Motion, which are framework for classical mechanics, including basis for modern engineering.

- **1st Law:** An object at rest, remains at rest unless acted on by an outside force. An object with uniform motion, remains in motion unless acted on by an outside force.

- **2nd Law:** An applied force, \( F \), on an object equals the rate of change of its momentum, \( p \), with time.

\[
\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt} (m\vec{v}) = \vec{v} \frac{dm}{dt} + m \frac{d\vec{v}}{dt}.
\]

- **3rd Law:** For every action there is an equal and opposite reaction.
Newton’s Theory of Gravity

Start with Kepler’s 3rd law:

\[ P^2 = kr^3 \]

For circular orbit:

\[ P = \frac{2\pi r}{v} \]

Insert this into Kepler’s 3rd law:

\[
\frac{4\pi^2 r^2}{v^2} = kr^3 \quad \rightarrow \quad \frac{4\pi^2}{k} = v^2 r
\]

Multiply both sides by factors:

\[
m \times \frac{1}{r^2} \times \frac{4\pi^2}{k} = (v^2 r) \times \frac{1}{r^2} \times m
\]

Yields:

\[
\frac{4\pi^2 m}{kr^2} = \frac{m v^2}{r} = F \quad \text{Force!}
\]
Newton’s Theory of Gravity

\[
\frac{4\pi^2 m}{kr^2} = m\frac{v^2}{r} = F
\]

By Newton’s 3rd law, this is the force on mass M from mass m, so the force on mass m should be:

\[
F = \frac{4\pi^2 M}{k'r^2}
\]

Equating these two formula gives:

\[
F = \frac{4\pi^2 Mm}{k''r^2}
\]

where \( k = k''/M \) and \( k' = k''/m \)

Defining \( G \equiv \frac{4\pi^2}{k''} \) we get the “usual” formula:

\[
F = \frac{G Mm}{r^2}
\]
Gravitational Acceleration

During the Apollo 15 mission, Commander David Scott does an experiment with a feather and hammer to see what happens:

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Derive Kepler’s Laws from Newton’s Laws

Define Center of Mass Frame

\[ \vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \]  
for 2 objects

\[ \vec{R} = \sum_{i=1\ldots N} m_i \vec{r}_i \]  
for \( i = 1 \ldots N \) objects

Differentiate with time,

\[ \vec{P} = M \vec{V} = \sum_{i=1 \ldots N} m_i \vec{v}_i = \sum_{i=1 \ldots N} \vec{p}_i \]

Differentiate with time, again:

\[ \frac{d\vec{P}}{dt} = \sum_{i=1 \ldots N} \frac{d\vec{p}_i}{dt} = \vec{F} = 0 \]  
by Newton’s 3rd Law
Choose Frame with $\dot{\mathbf{R}}=0$

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} = 0$$

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

$$\mathbf{r}_1 = - \left( \frac{m_2}{m_1 + m_2} \right) \mathbf{r}$$

$$\mathbf{r}_2 = + \left( \frac{m_1}{m_1 + m_2} \right) \mathbf{r}$$

Define “Reduced Mass” of the System: $\mu = \frac{m_1 m_2}{(m_1 + m_2)}$

$$\mathbf{r}_1 = - \frac{\mu}{m_1} \mathbf{r}$$

$$\mathbf{r}_2 = \frac{\mu}{m_2} \mathbf{r}$$
Derive Kepler’s Laws from Newton’s Laws

Now write out total Energy of the System

\[ E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - G \frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|} \]

Insert:

\[ \vec{r}' = \vec{r}_2 - \vec{r}_1 \]

And:

\[ \vec{r}_1 = -\frac{\mu}{m_1} \vec{r} \quad \rightarrow \quad v_1 = \frac{dr_1}{dt} = -\frac{\mu}{m_1} \frac{dr}{dt} \]

\[ \vec{r}_2 = \frac{\mu}{m_2} \vec{r} \quad \rightarrow \quad v_2 = \frac{dr_2}{dt} = \frac{\mu}{m_2} \frac{dr}{dt} \]

\[ \mu = \frac{m_1 m_2}{(m_1 + m_2)} = \frac{m_1 m_2}{M} \quad \rightarrow \quad m_1 m_2 = M \mu \]
Derive Kepler’s Laws from Newton’s Laws

Total Energy Becomes:

\[ E = \frac{1}{2} m_1 \left( -\frac{\mu}{m_1} v^2 \right) + \frac{1}{2} m_2 \left( \frac{\mu}{m_2} v^2 \right) - G \frac{M\mu}{r} \]

\[ E = \frac{1}{2} \frac{\mu^2}{m_1} v^2 + \frac{1}{2} \frac{\mu^2}{m_2} v^2 - G \frac{M\mu}{r} \]

Grouping Terms:

\[ E = \frac{1}{2} \mu^2 \left( \frac{1}{m_1} + \frac{1}{m_2} \right) v^2 - G \frac{M\mu}{r} \]

And:

\[ \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_2 + m_1}{m_1 m_2} = \frac{1}{\mu} \]

Which Gives:

\[ E = \frac{1}{2} \mu v^2 - G \frac{M\mu}{r} \]

Total Energy of the System is the sum of the Kinetic Energy of the reduced mass and the potential energy of the total mass/reduced mass system.
Introduce Angular Momentum:

\[ \vec{L} = \vec{r} \times \vec{p} = (\vec{r} \times m\vec{v}) \]

Look at rate-of-change of \( L \) as a function of \( t \):

\[
\frac{d\vec{L}}{dt} = \frac{d(\vec{r} \times \vec{p})}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times \vec{p} + \vec{r} \times \vec{F} = 0
\]

\[ = 0 \quad \text{, Conservation of Angular Momentum} \]
Similarly for the Angular Momentum:

$$\vec{L} = m_1\vec{r}_1 \times \vec{v}_1 + m_2\vec{r}_2 \times \vec{v}_2$$

Insert:

$$\vec{r}_1 = -\frac{\mu}{m_1}\vec{r} \quad \quad \vec{r}_2 = \frac{\mu}{m_2}\vec{r}$$

Leads to:

$$\vec{L} = -\mu\vec{r} \times \vec{v}_1 + \mu\vec{r} \times \vec{v}_2 = \mu\vec{r} \times (\vec{v}_2 - \vec{v}_1) = \vec{r} \times (\mu\vec{v}) = \vec{r} \times \vec{p}$$

The Two-Body problem may be treated as a One-body problem with $\mu$ moving about a fixed mass $M$ at a distance $r$. 
The Two-Body problem may be treated as a One-body problem with \( \mu \) moving about a fixed mass \( M \) at a distance \( r \).
Derive Kepler’s Laws from Newton’s Laws

\[
\vec{L} = \vec{r} \times \mu \vec{v} = \mu r \hat{r} \times \frac{d}{dt} (r \hat{r}) \\
= \mu r \hat{r} \times \left( \hat{r} \frac{dr}{dt} + r \frac{d\hat{r}}{dt} \right) = \mu r^2 \hat{r} \times \frac{d\hat{r}}{dt}
\]

\[
\vec{F} = - \left( \frac{GM \mu}{r^2} \right) \hat{r}
\]

\[
\vec{a} = - \left( \frac{GM}{r^2} \right) \hat{r}
\]

Take cross product:

\[
\vec{a} \times \vec{L} = - \frac{GM}{r^2} \hat{r} \times \left( \mu r^2 \hat{r} \times \frac{d\hat{r}}{dt} \right)
\]

\[
A \times (B \times C) = (A \cdot C) \cdot B - (A \cdot B) \cdot C
\]

\[
\vec{a} \times \vec{L} = -(GM\mu) \left[ \left( \hat{r} \cdot \frac{d\hat{r}}{dt} \right) \cdot \hat{r} - (\hat{r} \cdot \hat{r}) \frac{d\hat{r}}{dt} \right] = GM\mu \frac{d\hat{r}}{dt}
\]
Derive Kepler’s Laws from Newton’s Laws

\[ \overset{\rightarrow}{a} \times \overset{\rightarrow}{L} = GM\mu \frac{d\overset{\rightarrow}{r}}{dt} \]

\[ \frac{d}{dt} (\overset{\rightarrow}{v} \times \overset{\rightarrow}{L}) = \frac{d}{dt} (GM\mu \overset{\wedge}{r}) \quad \rightarrow \quad \overset{\rightarrow}{v} \times \overset{\rightarrow}{L} = GM\mu \overset{\wedge}{r} + \overset{\rightarrow}{D} \]

\[ \overset{\rightarrow}{D} \text{ is a constant} \]
\[ \overset{\rightarrow}{v} \times \overset{\rightarrow}{L} \text{ and } \overset{\wedge}{r} \text{ lie in the same plane, so must } \overset{\rightarrow}{D} \]
\[ D \text{ is directed toward Perihelion,} \]
\[ D \text{ determines } e \text{ (eccentricity) of ellipse.} \]
\[ \text{Maximum reached with } \overset{\wedge}{r} \text{ and } \overset{\rightarrow}{D} \text{ point in same direction.} \]
Derive Kepler’s Laws from Newton’s Laws

\( \vec{v} \times \vec{L} = GM\mu \hat{r} + \vec{D} \)

\( \vec{r} \cdot (\vec{v} \times \vec{L}) = (GM\mu \hat{r} + \vec{D}) \cdot \vec{r} \)

\[
A \cdot (B \times C) = (A \times B) \cdot C
\]

\[
(\vec{r} \times \vec{v}) \cdot \vec{L} = GM\mu r + rD \cos \theta
\]

\[
(\vec{r} \times \mu \vec{v}) = \vec{L} \quad \rightarrow \quad L^2/\mu = GM\mu r + rD \cos \theta
\]

\[
= GM\mu r \left(1 + \frac{D \cos \theta}{GM\mu}\right)
\]

Kepler’s 1st Law!

\[
r = \frac{L^2/\mu^2}{GM(1 + e \cos \theta)}
\]

\[
e = \frac{D}{GM\mu}
\]
Planets Follow Elliptical Orbits

The Inner Solar System

The Outer Solar System
Kepler’s First Law

Not just planets! Comets have elliptical orbits too.
Not just planets! Comets have elliptical orbits too.

**Kepler’s First Law**

![Comet Diagram]

**Halley’s Comet (76 yr orbit)**
Kepler’s First Law

Not just planets! Comets have elliptical orbits too.

Halley’s Comet (76 yr orbit)
Integrate from Focus to r:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

Let: $$\vec{v} = \vec{v}_r + \vec{v}_\theta$$

$$\frac{dr}{dt} \hat{r} + r \frac{d\theta}{dt} \hat{\theta} \rightarrow \frac{dA}{dt} = \frac{1}{2} r v_\theta$$

$$r$$ and $$v_\theta$$ are perpendicular, therefore:

$$r v_\theta = |r \times v| = \left| \frac{L}{\mu} \right| = \frac{L}{\mu}$$

$$\frac{dA}{dt} = \frac{L}{2\mu} = \text{Constant} \quad \text{Kepler’s 2nd Law}$$
Integrate 2nd Law with time: \[ A = \frac{1}{2} \frac{L}{\mu} \int dt \]

\[ A = \frac{1}{2} \left( \frac{L}{\mu} \right) P, \]

where \( P \) = period

\[ A = \pi ab \]

\[ P^2 = \frac{4\pi^2 a^2 b^2 \mu^2}{L^2} \]

1st Law: \[ r = \frac{L^2 / \mu^2}{GM (1 + e \cos \theta)} \rightarrow L = \mu \left[ GMa (1-e^2) \right]^{1/2} \]

Kepler’s 3rd Law \[ P^2 = \frac{4\pi^2 a^3}{G (m_1 + m_2)} \]
Calculate Orbital Speed at Perihelion and Aphelion

Use Kepler’s 1st Law

\[ r = \frac{L^2/\mu^2}{GM(1 + e \cos \theta)} \]

At Perihelion

\[ L_p = \mu v_p r_p \quad \cos \theta_p = 1.0 \]

At Aphelion

\[ L_a = \mu v_a r_a \quad \cos \theta_a = -1.0 \]

\[ r_p = \frac{(\mu v_p r_p)^2/\mu^2}{GM(1 + e)} \]

\[ v_p^2 = \frac{GM(1 + e)}{r_p} \]

\[ = \frac{GM(1 + e)}{a(1 - e)} \]

Total Energy:

\[ E = \frac{1}{2} \mu v^2 - \frac{GM \mu}{r} \]

\[ E = \frac{1}{2} \mu \left( \frac{GM(1 + e)}{a(1 - e)} \right) - \frac{GM \mu}{r_p} \]

\[ = \frac{GM \mu(1 + e)}{2r_p} - \frac{2GM \mu}{2r_p} \]

\[ = \frac{GM \mu(1 + e - 2)}{2r_p} = -\frac{GM \mu}{2a} \]
Total Energy in a Bound Orbit

\[ E_{\text{tot}} = \frac{-GM\mu}{2a} = \frac{-Gm_1m_2}{2a} = \frac{<U>}{2} \]

That is the total energy is 1/2 of the average potential energy. This is the Virial Theorem.

More in depth derivation by Rudolf Clausius (reproduced in book). It has many applications in astrophysics, including providing evidence for Dark Matter in Clusters of Galaxies!

Rudolf Julius Emanuel Clausius (January 2, 1822 – August 24, 1888), was a German physicist. He is one of the central founders of the science of thermodynamics.
What have we learned?

- Kepler’s Three Laws of Orbital Motion describes the orbit of all celestial bodies (planets, comets, etc. around the Sun).

- Newton’s theory of gravity combined with Conservation of Angular Momentum is physical basis for Kepler’s laws.

- Virial theorem states that objects in Bound Orbits satisfy the total energy equation

\[ E_{\text{tot}} = \frac{-GM\mu}{2a} = \frac{-Gm_1m_2}{2a} = \frac{<U>}{2} \]
Tidal Forces
What Causes Ocean Tides?
High Tide in Bay of Bundy

Low Tide
Saturn as seen by Spacecraft Cassini, October 2004
Tidal Forces

\[ F_{c,x} = -\frac{G M m}{r^2} \quad F_{c,y} = 0 \]
\[ F_{p,x} = \left(\frac{G M m}{s^2}\right) \cos\phi \quad F_{p,y} = -\left(\frac{G M m}{s^2}\right) \sin\phi \]

\[ \Delta \vec{F} = \vec{F}_p - \vec{F}_c = G M m \left( \frac{\cos\phi}{s^2} - \frac{1}{r^2} \right) \hat{\phi} - \left( \frac{G M m}{s^2} \right) \sin\phi \hat{\phi} \]

\[ s^2 = (r - R \cos\theta)^2 + (R \sin\theta)^2 = r^2 \left( 1 - \frac{2R \cos\theta}{r} \right) \quad R \ll r \]

\( M = \) mass of Moon; \( m = \) test mass
Tidal Forces

\[ \Delta F = F_p - F_c = G \frac{Mm}{s^2} \left( \frac{\cos \phi}{r^2} - \frac{1}{r^2} \right) \hat{i} - \left( \frac{G Mm}{s^2} \right) \sin \phi \hat{j} \]

\[ s^2 = (r - R \cos \theta)^2 + (R \sin \theta)^2 = r^2 \left( 1 - \frac{2R \cos \theta}{r} \right) \quad R \ll r \]

\[ \Delta F \approx G \frac{Mm}{r^2} \left[ \cos \phi \left( 1 + \frac{2R \cos \theta}{r} \right) - 1 \right] \hat{i} \]

\[ - G \frac{Mm}{r^2} \left[ 1 + \frac{2R \cos \theta}{r} \right] \sin \phi \hat{j} \]

For the Earth-Moon,

\[ \cos \phi \approx 1, \quad r \sin \phi = R \sin \theta, \quad \sin \phi = (R/r) \sin \theta \]

\[ \Delta F \approx G \frac{MmR}{r^3} \left[ 2 \cos \theta \hat{i} - \sin \theta \hat{j} \right] \]
Tidal Forces

\[ \Delta \vec{F} \approx G \frac{M m R}{r^3} \left[ 2 \cos \theta \hat{i} - \sin \theta \hat{j} \right] \]

Force of the Moon on the Earth

Differential (relative) force on the Earth, relative to the center
Tidal Forces

\[ \Delta F \approx G \frac{M_m R}{r^3} \left[ 2 \cos \theta \hat{i} - \sin \theta \hat{j} \right] \]

When will the tidal force equal the force of gravity? Consider the Earth-Moon system:

- \( M_m, R_m \) = mass of moon, radius of moon
- \( M_E, R_E \) = mass of Earth, radius of Earth

\[
G \frac{M_m m}{R_m^2} = \left( 2G M_E m / r^3 \right) R_m
\]

Assume constant density: \( M_m = \frac{4}{3} \pi R_m^3 \bar{\rho}_m \)

\[
r^3 = \left( 2M_E / M_m \right) R_m^3 = \left( 2R_E^3 \bar{\rho}_E \right) / \left( R_m^3 \bar{\rho}_m \right) R_m^3
\]
Tidal Forces

\[
G \frac{M_m m}{R_m^2} = \left( \frac{2GM_Em}{r^3} \right) R_m
\]

Assume constant density: \( M_m = \frac{4}{3} \pi R_m^3 \bar{\rho}_m \)

\[
r^3 = \left( \frac{2M_E}{M_m} \right) R_m^3 = \left( \frac{2R_E^3 \bar{\rho}_E}{R_m^3 \bar{\rho}_m} \right) R_m^3
\]

\[
r = \left( \frac{2\bar{\rho}_E}{\bar{\rho}_m} \right)^{\frac{1}{3}} R_E
\]

or

\[
r = f_R \left( \frac{\bar{\rho}_E}{\bar{\rho}_m} \right)^{\frac{1}{3}} R_E, \quad f_R = 2 \frac{1}{3} = 1.2599
\]

Roche, using a more sophisticated assumption for the density of the Moon calculated \( f_R = 2.456 \)
Comet Shoemaker-Levy 9 after passing through Jupiter’s Roche Limit
Saturn as seen by Spacecraft Cassini, October 2004
Planetary Moons form when a moon crosses the Roche Limit

Saturn

Rings

Janus
Epimetheus

Roche Limit

Mimas  Enceladas  Tethys

Dione

Rhea

Other Moons

Distance from Saturn Center in $R_S$

1  2  3  4  5  6  7  8

Friday, January 13, 2012
What have we learned?

- Tidal forces arise from differential gravitational force on the surface of a body.

- Magnitude of Tidal force is proportional to $r^{-3}$ where $r$ is the distance from the object to the source of the Tidal force.

- Tidal forces can cause moons to disintegrate as they cross a threshold where the strength of the Tidal force exceeds the binding strength of the moon (or any other object). This is the likely explanation for the formation of rings around giant planets.