Fundamental Cosmological Observations

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2. Faint galaxies are distributed uniformly over large angular scales (for faint galaxies, R > 20 mag).

3. With the exception of M31 (Andromeda), nearly all galaxies are moving away from us, with a recessional velocity much greater than the Milky Way’s escape velocity. Recessional velocity scales with distance (Hubble’s law).

4. Oldest stars and star clusters have ages ~12-13 Gyr.

5. Cosmic Microwave Background (CMB) is observed. It is isotropic (same in all directions) to 1 part in 100,000.

6. Spectrum of CMB is thermal, with $T_0 = 2.728 \pm 0.004$ K.

7. Cosmic abundance of Helium is 25-30%.

8. Number counts of radio galaxies does not follow $N(>S) \sim S^{-3/2}$. 
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In 1925 Edwin Hubble discovered Cepheid Variables in M31 (Andromeda “Nebula”). Hubble continued his search for Cepheids, and determined the distances to 18 galaxies.

At the same time, V. M. Slipher at Lowell Observatory looked at velocity shifts of extragalactic “nebulae” using the Calcium “HK” lines (Ca II, like in the Sun).

Vesto Slipher (1875-1969)
The Extragalactic Distance Scale

Radial velocities of nebulae measured by Slipher:

<table>
<thead>
<tr>
<th>NGC</th>
<th>velocity (km/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>221</td>
<td>-300</td>
</tr>
<tr>
<td>224</td>
<td>-300</td>
</tr>
<tr>
<td>598</td>
<td>~zero</td>
</tr>
<tr>
<td>1023</td>
<td>+200 roughly</td>
</tr>
<tr>
<td>1068</td>
<td>+1100</td>
</tr>
<tr>
<td>3031</td>
<td>+ small</td>
</tr>
<tr>
<td>3115</td>
<td>+400 roughly</td>
</tr>
<tr>
<td>3627</td>
<td>+500</td>
</tr>
<tr>
<td>4565</td>
<td>+1000</td>
</tr>
<tr>
<td>4594</td>
<td>+1100</td>
</tr>
<tr>
<td>4736</td>
<td>+200 roughly</td>
</tr>
<tr>
<td>4826</td>
<td>+ small</td>
</tr>
<tr>
<td>5194</td>
<td>+ small</td>
</tr>
<tr>
<td>5866</td>
<td>+600</td>
</tr>
<tr>
<td>7331</td>
<td>+300 roughly</td>
</tr>
</tbody>
</table>

Vesto Slipher (1875-1969)
The Extragalactic Distance Scale

In 1929, Hubble showed that the velocities and distances are linearly correlated, and satisfy

\[ v = H_0 \, d \]

where \( v \) is the recessional velocity (km/s) and \( d \) is the distance (Mpc). \( H_0 \) is a constant, “Hubble’s Constant” and has units of \( \text{km s}^{-1} \, \text{Mpc}^{-1} \).
The Extragalactic Distance Scale

Size of Grid x 1.01
The Extragalactic Distance Scale

Size of Grid x 1.02
The Extragalactic Distance Scale

Size of Grid x 1.03
Points the farthest away, also have moved the furthest.

Size of Grid $\times 1.04$
The Extragalactic Distance Scale
The Extragalactic Distance Scale

Size of Grid x 1.02
The Extragalactic Distance Scale

Size of Grid x 1.03
The Extragalactic Distance Scale

Size of Grid $\times 1.05$

Thursday, December 6, 2012
The Extragalactic Distance Scale

Size of Grid x 1.07
The Extragalactic Distance Scale

Size of Grid \times 1.10
The effect of doubling the size of the Earth, as viewed from Salt lake City
Cosmological Expansion

We will use this notation. At some time *today* (now), $t = t_0$.

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$a(t)$ is the expansion scale factor, defined to be $a(t_0) = 1$.

Distance between any two objects is then $r(t) = a(t) \times$. 
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Thursday, December 6, 2012
Color-magnitude diagram for Globular Clusters

M5

GC 47 Tuc

V [mag]

B-V color [mag]

Thursday, December 6, 2012
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Blackbody with $T=2.725$ K.


Penzias and Wilson received the Nobel Prize in 1978.
John Mather received the Nobel Prize in 2006
Global Projection of the Earth’s Map
Black Body $T=2.725$ K
Thermal component subtracted, $\Delta T=3.353$ mK
Dipole component subtracted, $\Delta T = 18 \, \mu K$
Origin of Structure

WMAP image

All components removed but background fluctuations
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BIG BANG NUCLEOSYNTHESIS

Why is ~25% of the mass in the Universe Helium (and the rest Hydrogen?)

At a time $t \sim 10^{-4}$ s the cosmic temperature was $\sim 10^{12}$ K. The Universe was filled with photons, electrons-positrons, neutrinos, and a few protons and neutrons.

There were $\sim 5$ p and n for every $10^{10}$ photons, therefore the former were constantly being transformed:

$$n \rightleftharpoons p^+ + e^- + \bar{\nu}_e$$
$$n + e^+ \rightleftharpoons p^+ + \bar{\nu}_e$$
$$n + \nu_e \rightleftharpoons p^+ + e^-$$

The mass difference between a p and n is only: $(m_p - m_n)c^2 = 1.293$ MeV.

The thermal energy of the Universe is $10^{12}$ K is $kT \sim 100$ MeV. At this energy, the # of neutrons and protons are nearly equal.
BIG BANG NUCLEOSYNTHESIS

As the Universe expanded, the temperature fell. At $T \sim 10^{10}$ K the number of neutrons to protons was “frozen” at 0.223. There were 223 neutrons for every 1000 protons, and no more neutrons were being created.

The half life of a free neutron is $614 \text{ s} = 10.2 \text{ min}$.

For $p + n$ to form deuterium requires $T \sim 10^9$ K (higher temperatures dissociate deuterium). In this time 176 s passed, so if you started with 223 neutrons, you’d now have 183, and number of protons increased to 1040.

Now deuterium and Helium fusion can occur at $T \sim 10^9$ K.

\[
\begin{align*}
\frac{2}{1}H + \frac{2}{1}H & \rightleftharpoons \frac{3}{1}H + \frac{1}{1}H \\
\frac{3}{1}H + \frac{2}{1}H & \rightleftharpoons \frac{4}{2}He + n
\end{align*}
\]

and

\[
\begin{align*}
\frac{2}{1}H + \frac{2}{1}H & \rightleftharpoons \frac{3}{2}He + n \\
\frac{3}{1}He + \frac{2}{1}H & \rightleftharpoons \frac{4}{2}He + \frac{1}{1}H
\end{align*}
\]

No other elements were made except trace amounts of Li-3. The 183 neutrons and 1040 protons could form 92 He-4 nuclei with 867 protons left over. Because He-4 is 4x more massive than Hydrogen, we should have: $\left[ \frac{4(92)}{(867 + 4(92))} \right] = 0.299$. 

Thursday, December 6, 2012

1. \( p \leftrightarrow n \)
2. \( p(n, \gamma)d \)
3. \( d(p, \gamma)^3\text{He} \)
4. \( d(d, n)^3\text{He} \)
5. \( d(d, p)t \)
6. \( t(d, n)^4\text{He} \)
7. \( t(\alpha, \gamma)^7\text{Li} \)
8. \( ^3\text{He}(n, p)t \)
9. \( ^3\text{He}(d, p)^4\text{He} \)
10. \( ^3\text{He}(\alpha, \gamma)^7\text{Be} \)
11. \( ^7\text{Li}(p, \alpha)^4\text{He} \)
12. \( ^7\text{Be}(n, p)^7\text{Li} \)

Current WMAP measurement (Komatsu 2009)
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Source Counts

Consider population of sources with Luminosity Function that is constant in space in time. \( n(>L) \) is the number density of sources with luminosity \( >L \).

The number of sources at a distance interval \( r \) and \( r+dr \) is

\[
n(>L) 4\pi r^2 dr
\]

The flux, \( S \), and luminosity are related by the distance \( L = 4\pi r^2 S \).

Therefore, the number of sources with flux \( >S \) in the interval \( r, r+dr \) is

\[
dN(>S) = 4\pi r^2 dr n(>4\pi r^2 S)
\]

Changing \( dr \) to \( dL \):

\[
r = \sqrt{L/(4\pi S)} \quad dr = dL/(2\sqrt{4\pi LS})
\]

\[
N(>S) = \int_0^\infty \frac{dL}{2\sqrt{4\pi LS}} \frac{L}{4\pi S} n(>L)
\]

\[
= \frac{1}{16\pi^{3/2}} S^{-3/2} \int_0^\infty dL \sqrt{L} n(>L)
\]

Therefore, if \( n(>L) \) is constant throughout the Universe, then \( N(>S) \sim S^{-3/2} \).
Universe cannot be Euclidean (flat), static, and infinite.
Equations of Cosmological Expansion

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$r(t_0) = a(t_0) \times x$
Equations of Cosmological Expansion

**Cosmological Principle:** The Universe is isotropic and homogeneous, appearing the same in all directions and at all locations.

To show the expansion is the same everywhere consider the following:

\[
\vec{v}_A = H_0 \vec{r}_A \\
\vec{v}_B = H_0 \vec{r}_B
\]

Doing some vector math:

\[
\vec{v}_B - \vec{v}_A = H_0 \vec{r}_B - H_0 \vec{r}_A = H_0 (\vec{r}_B - \vec{r}_A)
\]

In other words, the Hubble relation is exactly the same for an observer in galaxy A seeing B and observer in galaxy B seeing A. Expansion appears to be the same to all observers.
Simple Model of the Universe using Newtonian Physics

Consider a Spherical shell in a thin “dust” filled Universe. Dust is everywhere with a uniform density $\rho(t)$.

As the Universe expands, the dust is carried with it. Let $r(t)$ be the radius at time $t$ of a thin spherical shell containing mass $m$.

This shell expands with the Universe with recessional velocity $v(t) = \frac{dr(t)}{dt}$

$$v(t) = \frac{d}{dt}r(t) = \frac{da}{dt}x = \dot{a}x = \frac{\dot{a}}{a}r$$

$$\equiv H(t)r$$

$$H(t) \equiv \frac{\dot{a}}{a}$$
Equations of Cosmological Expansion

Let $M$ be the mass interior to the shell,

$$M(x) = \frac{4}{3} \pi x^3 \rho_0 = \frac{4}{3} \pi r^3(t) \rho(t)$$

Note that $M = \text{constant}$ since no mass is created, the volume expands in lock-step with the decrease in density. Density varies with time,

$$\rho(t) = \rho_0 a^{-3}(t)$$

Equation of motion for a particle on the spherical surface is

$$\ddot{r} = -\frac{GM(x)}{r^2} = -\frac{4\pi G}{3} \frac{\rho_0 x^3}{r^2}$$

Using $\ddot{a} = \ddot{r}/x$

$$\ddot{a}(t) = -\frac{4\pi G}{3} \frac{\rho_0}{a^2(t)} = -\frac{4\pi G}{3} \rho(t)a(t)$$
Equations of Cosmological Expansion

Also use Conservation of Energy, \( E = K(t) + U(t) = \text{constant} \).

To see this, multiply previous expression by

\[
2\dot{a} \times \ddot{a}(t) = -\frac{4\pi G}{3} \rho(t) a(t) \times 2\dot{a}
\]

Which gives, using the following

\[
\frac{d}{dt} \left( \dot{a}^2 \right) = -\frac{8\pi G}{3} \rho_0 \frac{\dot{a}}{a^2(t)} \quad \text{and} \quad \frac{d}{dt} \left( -\frac{1}{a} \right) = \frac{\dot{a}}{a^2}
\]

\[
\frac{d}{dt} \left( \dot{a}^2 \right) = \frac{8\pi G}{3} \rho_0 \frac{d}{dt} \left( \frac{1}{a} \right)
\]

\[
\dot{a}^2 = \frac{8\pi G}{3} \rho_0 \frac{1}{a} - Kc^2 = \frac{8\pi G}{3} \rho(t) a^2(t) - Kc^2
\]
Equations of Cosmological Expansion

\[ \dot{a}^2 = \frac{8\pi G}{3} \frac{1}{a} - Kc^2 = \frac{8\pi G}{3} \rho(t) a^2(t) - Kc^2 \]

This is the same as writing:

\[ \frac{v^2(t)}{2} - \frac{GM}{r(t)} = -Kc^2 \frac{x^2}{2} \]

The physical nature of the constant $K$ decides the fate of the Universe:

1. if $K > 0$ then the total energy of the shell is negative and the universe is **bounded** (closed). The expansion must someday halt and reverse.

2. If $K = 0$ then the total Energy is exactly zero. The expansion will continue for every and asymptote to zero recessional velocity. The Universe is **Flat**.

3. If $K < 0$ the total energy is positive and the universe will is **unbounded** (open). The Expansion will continue forever.
Equations of Cosmological Expansion

For the k=0, “flat” case, at t=t₀, a=1, and H₀ = da/dt  \[ H₀^2 = \frac{8\pi G}{3} \rho₀ \]

\[ \rho_c(t) = \frac{3H^2(t)}{8\pi G} \]

The present day value (t=t₀) is then:

\[ \rho_{c,0} = \frac{3H₀^2}{8\pi G} = 1.88 \times 10^{-26} h^2 \text{ kg m}^{-3} \]

Where H₀ = 100 h km/s/Mpc (h is the Hubble parameter). Our current measure is h=0.71 (more on this), which gives:

\[ \rho_{c,0} = 9.47 \times 10^{-27} \text{ kg m}^{-3} \]

Or about 6 Hydrogen atoms per cubic meter. Note that the best estimate of **baryonic matter density** is about 4% of the critical density, or 2 protons per 2 m x 2 m x 2m box!
The ratio of any density to the critical density is the density parameter, $\Omega$.

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G \rho(t)}{3H^2(t)}$$

The present day cosmic matter density is then

$$\Omega_0 = \frac{\rho_0}{\rho_{c,0}} = \frac{8\pi G \rho_0}{3H_0^2}$$

\[ [\Omega_{m,0}]_{\text{WMAP}} = (0.135^{+0.008}_{-0.009}) \, h^{-2} = 0.27 \pm 0.04 \text{ (for } h = 0.71) \]

$$\rho_{m,0} = 2.56 \times 10^{-27} \text{ kg m}^{-3} \text{ (for } h = 0.71)$$

\[ [\Omega_{b,0}]_{\text{WMAP}} = (0.0224 \pm 0.0009) \, h^{-2} = 0.044 \pm 0.004 \text{ (for } h = 0.71) \]
Equations of Cosmological Expansion

General characteristics of our Universe can be determined:

\[
\frac{\Omega}{\Omega_0} = \frac{\rho}{\rho_0} \frac{H_0^2}{H^2} = (1 + z)^3 \frac{H_0^2}{H^2} \quad \text{or} \quad \Omega H^2 = (1 + z)^3 \Omega_0 H_0^2
\]

We can insert our definition for \( \Omega \) into the above equations to get:

\[
H^2(1 - \Omega)R^2 = -kc^2 \quad \text{which for } t=t_0 \text{ becomes } H_0^2(1 - \Omega_0) = -kc^2
\]

Thus for \( \Omega_0 > 1 \) the Universe is \textbf{closed}.

For \( \Omega_0 = 1 \) the Universe if \textbf{flat}.

For \( \Omega_0 < 1 \) the Universe is \textbf{open}.

Equating the above relations and solving for \( \Omega \) gives:

\[
\Omega = \left( \frac{1 + z}{1 + \Omega_0 z} \right) \Omega_0 = 1 + \frac{\Omega_0 - 1}{1 + \Omega_0 z}
\]

This implies that as \( z \to \infty \) \( \Omega \to 1 \). The Universe would be very, very flat. This seemed too perfect; too good to be true to physicists....
So far we have ignored General Relativity (GR). GR would modify our interpretation by.

1. $E=mc^2$. We have to include this energy density in the equation of state of the Universe. This acts like matter that exerts a *Pressure*.

2. Einstein’s GR solution does not allow for a static Universe. Einstein had to add a Cosmological Constant to achieve a static Universe (to fit contemporary observations).

3. Interpretation of Expansion is not that objects are moving away from each other, but that *space* itself is expanding. This manifests itself as an expansion.
Equations of Cosmological Expansion

Consider the 1st law of thermodynamics. As “stuff” is compressed, it pushes back with a pressure. An adiabatic chance in volume $dV$ equals a change in internal energy:

$$dU = -P \, dV$$

for GR as applied to a homogeneous isotropic Universe, this gives

$$\frac{d}{dt} \left( c^2 \rho a^3 \right) = -P \frac{da^3}{dt}$$

where $\rho c^2$ is the energy density and $P$ is the pressure.

Considering a comoving volume element, $V_x$, the physical volume

$$V = a^3(t) \, V_x$$

says that $V/V_x = a^3$ and $c^2 \rho \, a^3$ is the energy density in the volume (divided by $V_x$).
Equations of Cosmological Expansion

GR includes the effects of pressure, which yields the **Friedman-Lemaître** Expansion Equations.

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{K c^2}{a^2} + \frac{\Lambda}{3}
\]

\[
\ddot{a} \frac{\dot{a}}{a} = -\frac{4\pi G}{3} \left( \rho + \frac{3P}{c^2} \right) + \frac{\Lambda}{3}
\]
Equations of Cosmological Expansion

Components of Matter in the Universe

For a gas moving in thermal motion at speeds much less than the speed of light, the pressure doesn’t matter. The thermal “speed” is the sound speed, $c_s$. For low $c_s$, $P \sim \rho c_s^2 \ll \rho c^2$. This is pressure-less matter, and assumes $P_m=0$.

In the case that velocities are comparable to the speed of light, then the pressure will not be non-negligible. (This is radiation.)

Same is true for particles with vanishing rest mass ($kT >> mc^2$).

For “thermal” objects, the pressure is

$$P_{\text{rad}} = \frac{1}{3} \rho_r c^2$$
Components of Matter in the Universe

Because the Vacuum energy density is constant (zero time derivative), then our formula:

\[
\frac{d}{dt} \left( c^2 \rho a^3 \right) = -P \frac{d a^3}{dt}
\]

Takes on the form:

\[
P_v = -\rho_v c^2
\]
Equations of Cosmological Expansion

Components of Matter in the Universe

\[
\frac{d}{dt} \left( c^2 \rho a^3 \right) = -P \frac{d a^3}{dt}
\]

We now can write the evolution of these components.

For pressureless matter, P=0, so:

\[
\rho_m(t) = \rho_{m,0} a^{-3}(t)
\]

For Radiation, \( P_r = (1/3) \rho c^2 \), so:

\[
\rho_r(t) = \rho_{r,0} a^{-4}(t)
\]

For the vacuum density, \( P_v = -\rho_v c^2 \), so: \( \rho_v = \text{const.} \)
Equations of Cosmological Expansion

\[ \rho_{c,0} = \frac{3H_0^2}{8\pi G} = 1.88 \times 10^{-29} \ h^2 \ g \ cm^{-3} \]

\[ \rho_{c,0} = 9.47 \times 10^{-30} \ kg \ cm^{-3} \quad \text{for } h=0.71 \]

\[ \Omega_m = \frac{\rho_m}{\rho_c} \]

\[ \Omega_r = \frac{\rho_r}{\rho_c} \quad \text{today: } \quad \Omega_r \sim 4 \times 10^{-5} \ h^{-2} \]

\[ \Omega_\Lambda = \frac{\rho_v}{\rho_c} = \frac{\Lambda}{3H_0^2} \]

\[ \Omega_0 = \Omega_m + \Omega_r + \Omega_\Lambda \]
Equations of Cosmological Expansion

Using \( H(t) = \frac{\dot{a}(t)}{a(t)} \) and \( \rho = \rho_m,0 a^{-3} + \rho_r,0 a^{-4} \)

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{Kc^2}{a^2} + \frac{\Lambda}{3}
\]

becomes (with a little algebra....)

\[
H^2(t) = H_0^2 \left[ a^{-4}(t)\Omega_r + a^{-3}(t)\Omega_m + a^{-2}(t) \frac{Kc^2}{H_0^2} + \Omega_\Lambda \right]
\]

Solving for \( K \) at \( H(t_0)=H_0 \) and \( a(t_0)=1 \) yields

\[
K = \left( \frac{H_0}{c} \right)^2 (\Omega_0 - 1) \approx \left( \frac{H_0}{c} \right)^2 (\Omega_m + \Omega_\Lambda - 1)
\]

\( K=0 \) corresponds to flat space

\( K > 0 : K^{-1/2} \) is the curvature radius of a sphere. Closed space

\( K < 0 : \) space is hyperbolic.
Equations of Cosmological Expansion

\[ H^2(t) = H_0^2 \left[ a^{-4}(t)\Omega_r + a^{-3}(t)\Omega_m + a^{-2}(t)(1 - \Omega_m - \Omega_\Lambda) + \Omega_\Lambda \right] \]
The Redshift is the change in wavelength:

\[ \lambda(a) = a \lambda_{\text{obs}} \]

where \( \lambda_{\text{obs}} \) is the wavelength at \( a=1 \) (*now*).

\[ (1 + z) = \frac{\lambda_{\text{obs}}}{\lambda_e} \]

\[ (1 + z) = \frac{1}{a} \]
Equations of Cosmological Expansion

\[ H^2(t) = H_0^2 \left[ a^{-4}(t)\Omega_r + a^{-3}(t)\Omega_m + a^{-2}(t)(1 - \Omega_m - \Omega_\Lambda) + \Omega_\Lambda \right] \]

This is often written as

\[ H(z) = H(0)E(z) = H_0E(z) \]

where \( H(z) \) is Hubble’s “constant” measured by a hypothetical observer at redshift \( z \). Therefore:

\[ E(z) = \sqrt{\Omega_r (1 + z)^4 + \Omega_m (1 + z)^3 + \Omega_\Lambda + \Omega_k (1 + z)^2} \]

with \( \Omega_k = (1 - \Omega_m - \Omega_\Lambda) \)

\( E(z) \) has the confusing name “Angular Size Distance” (see Peebles 1993), this function shows up a lot for computing cosmological quantities (times, distances) because it contains the evolution of Hubble’s Constant.
Deriving Cosmological Quantities

How much time elapses as a function of the scale factor, a?
The Lookback time is the time between two periods in time.

\[
\frac{\dot{a}}{a} = H(t)
\]

For example, the age of the Universe is integrated from a=0 to a=1.

\[
dt = da \times (da/dt)^{-1} = da/(aH)
\]

\[
t(a) = \frac{1}{H_0} \int_0^a da \left[ a^{-2}\Omega_r + a^{-1}\Omega_m + (1 - \Omega_m - \Omega_\Lambda) + a^2\Omega_\Lambda \right]^{-1/2}
\]

For the special case of \(\Omega_m=1, \Omega_\Lambda=0\), the integral is solvable analytically:

\[
t_0 = \frac{2}{3H_0} = 6.7 \times 10^9 \, h^{-1} \, \text{yr}
\]
Deriving Cosmological Quantities

How much time elapses as a function of the scale factor, $a$?

The Lookback time is the time between two periods in time.

$$\dot{a} = \frac{a}{H(t)}$$

More generally, you can compute the time between any two periods, $a_1$ and $a_2$, or $z_1$ and $z_2$

$$da = dz \times (1 + z)^{-1}$$

$$t = H_0^{-1} \int_{z_1}^{z_2} \frac{dz}{(1 + z)E(z)}$$

(See Hogg 1999, arXiv:9905116 for very good explanation of deriving cosmological distances)

from Hogg 1999