Ehrenfest Model

A. Imagine 2R balls numbered consecutively from 1 to 2R distributed in two Urns so at the beginning there are \( R+N \), \(-R \leq N \leq R\), balls in Urn 1.

B. Choose at random an integer between 1 and 2R and move ball, whose number has been drawn, from urn in which it is to other urn. Repeat S times and ask for Probability \( Q(R+N|R+m|S) \) that after S drawings there should be \( R+m \) balls in urn 1.
C. The average excess over $R$ of the number of balls in urn 1 is given by

$$\sum_{m=-R}^{R} m Q(R+n|R+m; s) = n(1 - \frac{1}{R})^s$$

which in the limit

$$R \to \infty, \frac{1}{R} \to 0, s \tau = t$$

($\tau =$ duration time for each drawing) gives

$$n e^{-\tau t} \text{ (Newton's law of cooling)}$$

D. Let $P(n1m; s)$ denote probability that after $s$ drawings $R+m$ balls will be observed for the first time in urn 1 if there were $R+n$ balls in that urn at the beginning. In particular $P(n1n; s)$ is the probability that the recurrence time of state “$n$” is $s \tau$. It can be shown

$$\sum_{s=1}^{\infty} P(n1n; s) = 1$$

Each state will recur with prob. = 1.
E. Let $\Theta_n =$ mean recurrence time

$$\Theta_n = \sum_{s=1}^{\infty} s \mathbb{E} P(n|n;s) = \mathbb{E} \frac{(R+n)! (R-n)! \cdot 2^n}{(2R)!}$$

tells us how long on the average one will have to wait for state to recur.

**Examples**

1) $R = 10,000$, $n = 10,000$, $\mathbb{E} = 1$ sec.

$$\Theta = 2^{20000} \text{ sec.} \approx 10^{6000} \text{ years!}$$

2) $n = 0$, $\Theta \approx 100 \sqrt{\pi} \text{ sec.} \approx 175 \text{ sec.}$