Experiment V
Motion of electrons in magnetic field and measurement of e/m

In Experiment IV you observed the quantization of charge on a microscopic bead and measured the charge on a single electron. In this experiment you will observe the motion of electrons as they are first accelerated in an electric field and then follow a circular trajectory in a region of magnetic field.

The method is similar to that used by J.J. Thomson in 1897. A beam of electrons is accelerated through a known potential, so using the work-energy theorem the velocity of the electrons is known. A solenoid or a pair of Helmholtz coils produces an approximately uniform and measurable magnetic field in the region traversed by the electron beam. Depending upon the orientation of the magnetic field and the electron velocity, the electron beam will follow either a circular trajectory (if \( \vec{v} \perp \vec{B} \)) or a spiral trajectory (if \( \vec{v} \) has a component parallel to \( \vec{B} \)). By measuring the accelerating potential (\( V \)), the current in the magnet coils (\( i \)), and the radius of curvature of the electron beam trajectory (\( r \)), the ratio of charge to mass of the electron, \( e/m \), can be calculated.

When an electron (charge \( q = -e \)) travels with velocity \( \vec{v} \) in a magnetic field \( \vec{B} \), it experiences a Lorentz force

\[
\vec{F} = q\vec{v} \times \vec{B}
\]  

(1)

The Lorentz force is at right angles to both the direction of motion (\( \vec{v} \)) and the magnetic field. You have encountered this kind of force before in your study of mechanics: the central force exerted by the gravitation of the Sun acting upon the Earth causes the Earth to move in an orbit around the sun. When the orbit is circular, the gravitational force is always at right angles to the direction of the Earth’s motion at that instant. In such a case, the acceleration produced by the force cannot change the magnitude of the velocity vector (it has no component parallel to the velocity vector), it only changes the direction of the velocity vector.

This property is always true of constant (not changing in time) magnetic fields. A constant magnetic field cannot change the magnitude of the velocity of motion of a charged particle, it can only change the direction of motion. Note that any component of the vector magnetic field that is parallel to the particle’s velocity vector exerts no force at all on the particle.

To analyze the motion quantitatively, we use the same technique as in the case of orbital motion. We equate the centripetal force needed to maintain circular motion on a radius \( r \) to the Lorentz force produced by the magnetic field due to the component \( v_\perp \) of velocity that is perpendicular to the magnetic field \( \vec{B} \):

\[
\frac{mv_\perp^2}{r} = qvB
\]

\[
r = \frac{mv_\perp}{qB}
\]  

(2)

The radius of the orbit is called the gyro-radius.

If in addition there is a component \( v_\parallel \) of the magnetic field that is parallel to the electron velocity, it is a constant of motion (the velocity component in that direction does not change with time due to the magnetic field). In that case the resulting motion will be
a spiral, which is a superposition of the circular motion in the plane perpendicular to \( \vec{B} \) and uniform motion parallel to \( \vec{B} \).

In order to study the motion of charged particles in a magnetic field, we need a source of charged particles and means of giving them an initial velocity \( \vec{v} \), and a source of magnetic field \( \vec{B} \), and a way to visualize the particles. The source of charged particles is an electron gun similar to the one that you used in Experiment III. By accelerating the electrons through a potential \( V \), they attain a velocity \( v \):

\[
e V = \frac{1}{2} m v^2
\]

\[
v = \sqrt{2eV/m}
\]  \hspace{1cm} (3)

From Equations (2) and (3), it is evident that if you know the accelerating potential \( V \) and the magnetic field \( B \) and measure the gyroradius \( r \), you can obtain a measurement of the ration \( e/m \) of the charge to mass of the electron. This experiment provides a way for you to experimentally measure this ratio of properties of the point-like particle of nature.

You can measure \( V \) with a voltmeter, and \( r \) with a meter stick. You will learn how to measure \( \vec{B} \) using a Hall probe magnetometer. You will use two different geometries for your measurements. In the first geometry, you will insert the same cathode ray tube that you used in Experiment III into a solenoid, so that the entire electron beam trajectory is immersed in a magnetic field parallel to the tube axis. In the second geometry, you will use a spherical electron tube that is mounted within a Helmholtz pair of coils, so that the beam traverses a region of approximately uniform field in a fashion that enables you to actually observe the entire trajectory.
Problem 1
Electron trajectories in a cathode ray tube with axial magnetic field

You will need your lab report from Experiment III, in which you studied the deflection of an electron beam by a transverse electric field produced by a voltage $V_a$ across a pair of deflection plates. You will be using the same cathode ray tube that you used in that experiment. You will place it inside a solenoid coil that will be used to create an approximately uniform magnetic field throughout the electron tube. The tube is mounted in a support that aligns it so that the axis of the tube is parallel to the axis of the solenoid.

If the electron beam were to pass through the tube with no voltage applied to the deflection plates, its velocity would be perpendicular to the magnetic field and there would be no magnetic force upon the beam. If however a voltage $V_a$ is applied to either set of deflection plates, the deflection will produce a transverse component of velocity, and the electrons will experience a magnetic force.

The electron velocity produced by its acceleration in the electron gun is parallel to the magnetic field. Let’s define the $z$ axis to point along the common axis of the solenoid and the cathode ray tube. The velocity in that direction is given by Eqn. (3):

$$v_z = \sqrt{2eV_a/m}$$  \hspace{1cm} (4)

Now suppose that you apply a voltage across a pair of deflection plates that have a length $\ell$ and are separated in the $x$ direction by a transverse gap $w$. There will be an electric field $\vec{E} = \frac{V_a}{w} \hat{x}$, which will produce an acceleration $\vec{a} = \frac{e\vec{E}}{m}$ in the aperture between the deflection plates. As an electron passes through the gap, it will experience this acceleration for a time $T = \frac{\ell}{v_z}$, so that as it leaves the deflection plates it will have a transverse velocity $\vec{v}_0 = aT\hat{x}$.

Now as the electrons continue along the tube, they will experience a Lorentz force that causes them to follow a spiral trajectory. The following is a setup and solution of the equations of motion for the spiral trajectory.
\[
m \frac{d\vec{v}}{dt} = (-e)\vec{v} \times \vec{B}
\]

\[
m \frac{dv_z}{dt} = 0 \quad v_z = \text{const.}
\]

\[
\vec{v}_\perp = v_x \hat{x} + v_y \hat{y}
\]

\[
m \frac{d\vec{v}_\perp}{dt} = (-e)\vec{v}_\perp \times \vec{B}
\]

\[
\frac{dv_x}{dt} = -\frac{eB}{m} v_y
\]

\[
\frac{dv_y}{dt} = +\frac{eB}{m} v_x
\]

\[
\frac{d^2v_x}{dt^2} = -\frac{eB}{m} \frac{dv_y}{dt} = -\frac{eB}{m} \left( +\frac{eB}{m} v_x \right)
\]

\[
v_x = v_0 \cos \omega t
\]

\[
v_y = v_0 \sin \omega t
\]

\[
\omega = \frac{eB}{m}
\]

(5)

So the motion is a superposition of uniform motion in the z direction and circular motion in the x-y plane, hence a spiral.

We can integrate these equations of motion to obtain the radius of the circular motion and the pitch length with which the spiral makes a complete turn:

\[
x(t) = \int v_x(t) dt = -\frac{v_0}{\omega} \sin \omega t
\]

\[
y(t) = \int v_y(t) dt = +\frac{v_0}{\omega} \cos \omega t
\]

\[
z(t) = v_z t
\]

(6)

Inquiry

You should develop a strategy to use the spiral motion of the electron beam in the CRT to make a precision measure of the charge to mass ratio of the electron. You will first need to develop an expression for the radius \( R \) of spiral motion and the pitch length \( L \) with which the spiral repeats itself. You have control over the accelerating voltage \( V_a \), the deflecting voltage \( V_d \), and the current \( I \) in the solenoid.

You should measure the turns per unit length on the solenoid coil, and use Ampere’s Law to calculate the magnetic field in the solenoid as a function of current in the coil. You will use a Hall effect magnetometer to actually measure the field strength as a function of current in the solenoid and compare with your calculations.

Note: A Hall effect magnetometer measures the component of the vector magnetic field that is perpendicular to the face of the probe. You will learn in lectures this week about the Hall effect and how it is used to measure magnetic field. Be sure that you design a way to support your probe inside the solenoid so that its face is aligned as precisely as possible perpendicular to the solenoid axis.
It will help to use your lab report from Experiment III and your notes from that experiment for the geometry of the CRT and the analysis of the beam motion.

You cannot visualize the spiral trajectory directly inside the vacuum of the CRT. Your only observable is the location of the spot where the electron beam impacts the front face of the CRT. You will need to develop a technique by which you can measure the x-y location of that spot as a function of $V_a$, $V_d$, and $B$ and extract $e/m$ from those measurements. You should record your measurements in an EXCEL spreadsheet and use the spreadsheet to analyze the dependence of the beam spot on the experimental variables, plot your results appropriately, and obtain quantitative results, and estimate the magnitude of errors in your measurements and the effect of those errors on the accuracy of your measurement of $e/m$.

As an additional element of inquiry, keep in mind that the Earth’s magnetic field is not negligible in comparison with that of the solenoid. Devise a method to measure its effect and to eliminate errors that would result from its effect in your measurement of $e/m$.  

Problem 2
Measuring e/m in an open geometry Helmholtz pair

In Problem 1 you observed the spiral motion of an electron beam in a magnetic field. You were not able to actually see the electrons as they moved along their trajectories, only the bright spot on the fluor screen where they struck it. Now you will study the circular orbits of electrons moving in a field that is perpendicular to their motion. And you will be able to actually see the trajectory of the beam, because instead of providing a hard vacuum inside the electron tube we will provide a low pressure of helium gas.

Fluorescence from residual gas – visualizing the electron trajectories

When an electron collides with an atom of helium gas, it can excite or ionize an orbital electron. The ionization energy of helium is its ionization potential $E_0 = 24eV$, and the energy of the first excited state is $\frac{3}{4}E_0 = 18eV$, so the beam electron will lose that much energy in ionizing or exciting an orbital electron. Providing that the density of helium gas in the tube is low enough that on the average a beam electron makes only one such collision in its trajectory, and so long as the kinetic energy of the electrons in the beam is much larger than this ionization energy, a beam electron will lose only a modest fraction of its kinetic energy and so it will follow nearly the same trajectory that it would have followed with no collision.

When an electron is ionized from an atom or excited to the first excited state, it can be recaptured to the ground state while emitting a photon of light. This process of emission of light by transitions between allowed states in an atom is called fluorescence. It is described by the laws of quantum mechanics, which you will learn in a subsequent course. The light emitted by the excited atom makes it possible for you to see the location where the ionization occurred, and the emissions from atoms excited by the many electrons in the beam produce a nearly continuous luminous trail showing the trajectory of the electron beam.

You should calculate the pressure of helium gas that is needed to produce an average of one electron collision in the path length traversed by each electron traveling with velocity $v$ in the apparatus of the experiment. You will need to know the relationship between the density of helium gas and its pressure, which is given by the ideal gas law

$$P = nkT \quad (7)$$

Here $P$ is the pressure (in Pascals, the SI unit of measure), $n$ is the density of atoms ($m^{-3}$), $k = 1.38 \times 10^{-23} \text{ J/K}$ is Boltzmann’s constant. You will also need to know the cross-section $\sigma$ that measures the probability that an electron will ionize or excite an atom as it passes it. You can assume that this cross-section is just the size of the atom, $\sim \text{Å}^2$.

You can then estimate the scatterings per unit length as

$$dN/dl = n\sigma v$$

Estimate this pressure and translate it to units of mmHg – atmospheric pressure corresponds to 760 mmHg, the column height of a column of liquid Hg that would be sustained by one atmosphere of pressure. One atmosphere of pressure in English units is 14.5 pounds per square inch (psi), and you can convert this into SI units in order to obtain your pressure in mmHg. Compare your answer with that mentioned in the following text.
The Helmholtz Pair: Producing uniform magnetic field in an open geometry

It would be difficult to see the light from the circular trajectory of the electron beam if we had to place it inside a long solenoid as you did for Problem 1. Instead you will use a Helmholtz Pair: a pair of identical wire coils, oriented along a common axis, with the distance between them optimized to produce an approximately uniform magnetic field inside.

Magnetic field is produced by circulating currents of electric charge. To make such a circulating current in the lab, you can wind a piece of conducting wire into a circular coil of N turns. In this problem, you will design the placement two such coils to produce an approximately uniform field over a volume enclosed within the coil pair.

Just as the pattern of electric field produced in the space around a distribution of charge is not analytically calculable except for a few simple cases, that same is true for the pattern of magnetic field due to a pattern of current-carrying wire elements.

You have learned Ampere’s Law that relates the line integral of magnetic field around a closed path to the current $I$ flowing through the path:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \quad (8)$$

Ampere’s Law does not help you to calculate $\vec{B}$ at any given point on such a path, however, unless the integrand has the same value at all points along the path (the case with solenoids and infinite current sheets). We will be investigating the distribution of magnetic field produced by a single circular loop of wire of radius $R$, and by a pair of such loops spaced apart along a common axis. In such cases we will need to use the differential form of Ampere’s Law, called the Law of Biot and Savart. This law enables you to calculate the vector magnetic field $d\vec{B}$ produced by a current element of length $d\vec{l}$ and current $I$, at a location displaced $\vec{r}$ from the current element:

$$d\vec{B} = \frac{\mu_0 I}{4\pi^3} d\vec{l} \times \vec{r} \quad (9)$$

Use the Law of Biot and Savart to calculate an expression for the magnetic field produced by a single current loop of radius $R$, at a location on the axis of the loop a distance $z$ from the center of the loop. Then use this expression to calculate the field at the center (marked with the red asterisk) in the following geometry, in which you locate two such loops, each with its axis on the $z$ axis and carrying current in the same direction, at positions $z = \pm L/2$. You should be able to derive this expression as a function of the ratio $L/R$ that characterizes the coil geometry.
Now calculate the derivative of $B$ with respect to a small displacement $dz$ in this geometry. Note that to do this you will have to re-write the expression assuming that the field point is not at $z=0$ but at this displaced location $z$, include this displacement in the contributions of field from each coil, then take the derivative $dB_z/dz$, and finally set $z=0$.

Now you should find that $dB_z/dz = 0$ for any choice of $R/L$. That is just a consequence of the symmetry of the coil distribution around the x/y midplane.

Now take the second derivative $d^2B_z/dz^2$. Now you should obtain a non-zero result, but you should find that for one particular value of $R/L$ this second derivative will vanish! That is the design criterion for a Helmholtz pair: by placing the pair with this magic ratio $R/L$, the region of space inside the coil has an approximately homogeneous field strength, so you can locate an experiment in this region and have an approximately uniform magnetic field in an open geometry.

Measure the magnetic field at a point in the Helmholtz pair provided for your experiment that is accessible to place your Hall probe magnetometer. Compare your measurement with your calculations.

**The e/m apparatus in a Helmholtz pair**

The e/m apparatus also has deflection plates that can be used to demonstrate the effect of an electric field on the electron beam. This can be used as a confirmation of the negative charge of the electron, and also to demonstrate how an oscilloscope works.

One feature of the e/m tube is that the socket rotates, allowing the electron beam to be oriented at any angle (from 0-90 degrees) with respect to the magnetic field from the Helmholtz coils. You can therefore rotate the tube and examine the vector nature of the magnetic forces on moving charged particles.

**The e/m Tube**—The e/m tube (see Figure 2) is filled with helium gas at a pressure of $10^1$ mm Hg, and contains an electron gun and deflection plates. The electron beam
leaves a visible trail in the tube, because some of the electrons collide with helium atoms, which are excited and then radiate visible light.

The electron gun is shown in Figure 3. The heater heats the cathode, which emits electrons. The electrons are accelerated by a potential applied between the cathode and the anode. The grid is held positive with respect to the cathode and negative with respect to the anode. It helps to focus the electron beam.

Figure 1 The e/m Apparatus

CAUTION: The voltage to the heater of the electron gun should NEVER exceed 6.3 volts. Higher voltages will burn out the filament and destroy the e/m tube.
The Helmholtz Coils—The geometry of Helmholtz coils—the radius of the coils is equal to their separation—provides a highly uniform magnetic field. The Helmholtz coils of the e/m apparatus have a radius and separation of 15 cm. Each coil has 130 turns.

The Controls—The control panel of the e/m apparatus is straightforward. All connections are labeled. The hook-ups and operation are explained in the next section.

Cloth Hood—The hood can be placed over the top of the e/m apparatus so the experiment can be performed in a lighted room.

Mirrored Scale—A mirrored scale is attached to the back of the rear Helmholtz coil. It is illuminated by lights that light automatically when the heater of the electron gun is powered. By lining the electron beam up with its image in the mirrored scale, you can measure the radius of the beam path without parallax error.

Additional Equipment Needed—
Power Supplies:
- 6-9 VDC @ 3 A (ripple < 1%) for Helmholtz coils
- 6.3 VDC or VAC for filament
- 150-300 VDC accelerating potential
Meters:
- Ammeter with 0-2 A range to measure current in Helmholtz coils
- Voltmeter with 0-300 V range to measure accelerating potential
**Measuring electron trajectories**

1. If you will be working in a lighted room, place the hood over the e/m apparatus.

2. Flip the toggle switch up to the e/m MEASURE position.

3. Turn the current adjust knob for the Helmholtz coils to the OFF position.

4. Connect your power supplies and meters to the front panel of the e/m apparatus, as shown in Figure 4.

5. Adjust the power supplies to the following levels:
   
   ELECTRON GUN
   
   Heater: 6.3 VAC or VDC
   
   Electrodes: 150 to 300 VDC
   
   Helmholtz Coils: 6-9 VDC (ripple should be less than 1%)

   **CAUTION:** The voltage to the heater of the electron gun should NEVER exceed 6.3 volts. Higher voltages will burn out the filament and destroy the e/m tube.

6. Slowly turn the current adjust knob for the Helmholtz coils clockwise. Watch the ammeter and take care that the current does not exceed 2 A.

7. Wait several minutes for the cathode to heat up. When it does, you will see the electron beam emerge from the electron gun and it will be curved by the field from the Helmholtz coils. Check that the electron beam is parallel to the Helmholtz coils. If it is not, turn the tube until it is. Don’t take it out of its socket. As you rotate the tube, the socket will turn.

8. Carefully read the current to the Helmholtz coils from your ammeter and the accelerating voltage from your voltmeter. Record the values.

9. Carefully measure the radius of the electron beam. Look through the tube at the electron beam. To avoid parallax errors, move your head to align the electron beam with the reflection of the beam that you can see on the mirrored scale. Measure the radius of the beam as you see it on both sides of the scale, then average the results.
From your results calculate your best estimate of the charge-to-mass ratio, and compare that result with the one you obtained in Problem 1.