Physics 218 Honors
Exam 3 Solutions

1. You want to set a can of soda on an incline without it toppling over. The empty can has a mass of 100 g, the contents of a full can have a mass of 400 g. In order to stabilize the can on the incline, you decide to drink down the soda to a level that the center of mass of the partially full can is at its lowest level. What fraction of the contents should you drink?

Let \( m = 0.1 \text{ kg} \) = mass of empty can, \( M = 0.4 \text{ kg} \) = mass of contents, \( L \) = height of the top of the can.

Suppose you drink a fraction \( x \) of the contents. The center of mass of the un-drunk contents is then \( xL/2 \), and its mass is \( xM \).

The center of mass of the empty can is \( L/2 \).

So the center of mass of the partially full can is

\[
Y = \frac{(xM)(xL/2) + (m)(L/2)}{xM + m} = \frac{L}{2} \left( \frac{x^2 + \frac{m}{M}}{x + \frac{m}{M}} \right)
\]

Now you want to choose \( x \) so that the center of gravity is minimum:

\[
\frac{dY}{dx} = \frac{L}{2} \left( \frac{2x\left( \frac{m}{M} \right) - \left( \frac{x^2 + \frac{m}{M}}{x + \frac{m}{M}} \right)}{\left( x + \frac{m}{M} \right)^2} \right) = 0
\]

\[
x^2 + 2\frac{m}{M}x - \frac{m}{M} = 0
\]

Now we use the mass ratio \( m/M = 0.25 \):

\[
x^2 + \frac{x}{2} - \frac{1}{4} = 0
\]

\[
x = \frac{-\frac{1}{2} \pm \sqrt{\left( \frac{1}{2} \right)^2 + 1}}{2} = 0.31
\]

Note that I use only the positive root because \( x<0 \) is inconsistent with the assumptions of the problem.

So the fraction you should drink is \( 1-x = 0.69 \).
2. A sawhorse is constructed of 5 identical boards as shown. Each board has a length of 1 m. The angle between each pair of legs is 60°. How high above the base is the center of mass? If the sawhorse were set up on an inclined plane as shown, what is the maximum tilt angle θ for which the sawhorse would not topple?

We want the angle θ for which the center of mass is exactly over the pivot axis (the line along which the two boards contact the floor).

By symmetry, the center of mass of the sawhorse lies in the bisector plane. The center of mass of each board is at the center of its length L. We will now calculate the y coordinate of the center of mass of each board – the two boards forming the left legs (1), the two boards forming the right legs (2), and the cross-board (3):

\[ y_1 = y_2 = \frac{L}{2} \sin 60° \]
\[ y_3 = L \sin 60° \]
\[ Y = \frac{4 \left( \frac{L}{2} \sin 60° \right) + L \sin 60°}{5} = \frac{3\sqrt{3}}{10} L = 0.52m \]
\[ X = \frac{L}{2} \left( \frac{1}{2} + \cos 60° \right) = \frac{L}{2} \]

We now rotate the sawhorse by an angle θ so that the center of gravity is over the pivot:

\[ \tan \theta = \frac{X}{Y} = \frac{5}{3\sqrt{3}} \]
\[ \theta = 44° \]
3. A 100 kg box is transported at a speed of 1.2 m/s on a conveyor belt through a factory. It travels 1) up a ramp at an angle of 20° above horizontal; 2) along a horizontal stretch, and finally 3) down a ramp at an angle of 15° below horizontal. Calculate the power delivered by the conveyor drive to move the box along each of the three segments.

1) \[ P = F \cdot \dot{v} = (100 \text{ kg})(9.8 \text{ m/s}^2)(1.2 \text{ m/s})\sin 20° = 402W \]

2) \[ P = F \cdot \dot{v} = 0 \]

3) \[ P = F \cdot \dot{v} = (100 \text{ kg})(9.8 \text{ m/s}^2)(1.2 \text{ m/s})\sin(-15°) = -304W \]
4. A block of mass 12 kg falls from a height of 2 m onto a spring. The block sticks to the spring upon contact so that it cannot release from it on rebound. The spring compresses by a maximum amount of 0.3 m, at which time the block has come to a momentary rest. A) What is the spring constant of the spring? B) How high above the spring’s equilibrium point is the maximum stretch of the spring as the block rebounds?

A) The block stops when all of its gain in gravitational potential energy is converted into spring potential energy:

\[ mg(2 + 0.3) = \frac{1}{2} k(0.3)^2 \]

\[ k = \frac{2 \cdot (12 \text{ kg}) \cdot (9.8 \text{ m} / \text{s}^2)(2.3 \text{ m})}{(0.3 \text{ m})^2} = 6010 \text{ N/m} \]

B) When the spring rebounds, it stretches the spring beyond its equilibrium length. The block stops when the gain in potential energy equals the loss in spring potential energy:

\[ mg(2 - x) = \frac{1}{2} kx^2 \]

\[ 3005x^2 + (12)(9.8)x - 2(12)(9.8) = 0 \]

\[ x = \frac{-118 \pm \sqrt{118^2 + 4 \cdot 3005 \cdot 236}}{6010} = 0.26m \]
5. Tarzan wants Jane. Tarzan has a mass of 150 kg. He swings from a massless vine that is 25 m long, starting at the top of a cliff. At the bottom of the arc of his swing he descends 12 m in altitude. Jane waits on a bluff that is 6 m higher than the bottom of his swing. Unfortunately, the vine is not very strong. It will break if the tension exceeds 2500 N. Does Tarzan make it to Jane? If so, what is the maximum tension in the vine during his trip? If not, at what angle in his swing does the vine break?

Maximum tension in the rope would occur at the bottom of the arc:

\[
T - mg = \frac{mv^2}{R}
\]

\[
\frac{1}{2}mv^2 = mg\Delta y
\]

\[
T = mg\left[1 + 2\frac{\Delta y}{R}\right] = (150\text{kg})(9.8)(1 + 2\frac{12m}{25m}) = 2881N
\]

Alas, the vine will break.

The tension when the vine is at an angle \(\theta\) from the vertical is found from the free body diagram of the balance of forces parallel to the rope:

\[
T - mg \cos \theta = \frac{mv^2}{R}
\]

\[
\frac{1}{2}mv^2 = mg\left[\Delta y - R(1 - \cos \theta)\right]
\]

\[
T = mg\left[\cos \theta + 2\frac{\Delta y}{R} - 2(1 - \cos \theta)\right]
\]

\[
T = mg\left[3\cos \theta - 2 + 2\frac{\Delta y}{R}\right]
\]

We now set \(T=2500N\) to find out when the vine breaks:

\[
\cos \theta = \frac{1}{3}\left[\frac{T}{mg} + 2 - 2\frac{\Delta y}{R}\right] = \frac{1}{3}\left[\frac{2500N}{(150\text{kg})(9.8m/s^2)} + 2 - 2\frac{12}{25}\right] = 0.913
\]

\[
\theta = 24^\circ
\]
6. A 10 kg block slides along a horizontal surface with kinetic friction coefficient 0.25. When it is 60 cm from the end of a spring (spring constant 230 N/m), it is traveling with a velocity of 2.5 m/s towards the spring. 

A) What is the maximum compression of the spring? 

B) After the block recoils from the spring, at what distance from the end of the spring does the block come to a final stop? 

C) How much heat energy is generated by friction during the block’s travel?

A) Define $x=0$ to be the equilibrium end of the spring, and positive $x$ compresses the spring. To calculate the position $x_1$ where it stops the first time, use conservation of energy:

<table>
<thead>
<tr>
<th>initial</th>
<th>final</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$U$</td>
</tr>
<tr>
<td>$\frac{1}{2}mv_0^2$</td>
<td>0</td>
</tr>
</tbody>
</table>

$T_i = 31.25 J$

$115x_1^2 + 24.5x_1 - 31.25 + 14.7 = 0$

$x_1 = 0.29 m$

B) Now restart the problem, and calculate conservation of energy from this moment until the block stops the second time a distance $x_2$ to the left of the spring equilibrium position:

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$U$</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{1}{2}kx_1^2$</td>
</tr>
</tbody>
</table>

$\frac{1}{2}kx_1^2 = \mu mg(x_1 + x_2)$

$\frac{1}{2}kx_1^2 = 9.67 J$

$\mu mgx_1 = 7.1 J$

$x_2 = \frac{9.67 - 7.1}{0.25 \cdot 0.98} = 0.10 m$

C) All the initial KE is converted into heat:

$Q = \frac{1}{2}mv_0^2 = 31.25 J$