Solution for Practice of midterm exam

Problem 1:
An electron is held in a finite square potential well of width 1 Å. For which values of the well’s depth $V_0$ are there exactly two possible bound stationary states for the electron? (4pts)

Solution

We write the potential as

$$V = \begin{cases} 
0, & x < -a/2 \text{ Region I} \\
-V_0, & -a/2 < x < a/2 \text{ Region II} \\
0, & x > a/2 \text{ Region III} 
\end{cases}$$

Let’s define

$$\rho = \sqrt{-\frac{2mE}{\hbar^2}}, \quad k = \sqrt{\frac{2m(E + V_0)}{\hbar^2}}$$

Then the stationary solution for this region is:

$$\begin{align*}
\phi_I(x) &= Ae^{\rho x} \\
\phi_{II}(x) &= Be^{ikx} + B'e^{-ikx} \\
\phi_{III}(x) &= Ce^{-\rho x}
\end{align*}$$

Where we have used the condition that wave function is zero when we go to infinity. Now as $\phi(x)$ and $\phi'(x)$ are continue at $x = -a/2$ and $x = a/2$. we get

$$Ae^{-\rho a/2} = Be^{-ika/2} + B'e^{ika/2}, \quad \rho Ae^{-\rho a/2} = ikBe^{-ika/2} - ikB'e^{ika/2}$$

$$Ce^{-\rho a/2} = Be^{ika/2} + B'e^{-ika/2}, \quad -\rho Ce^{-\rho a/2} = ikBe^{ika/2} - ikB'e^{-ika/2}$$

We get

$$C = \left(\frac{\rho + ik}{2ik}e^{ika} - \frac{\rho - ik}{2ik}e^{-ika}\right)A, \quad -\frac{\rho}{ik}C = \left(\frac{\rho + ik}{2ik}e^{ika} + \frac{\rho - ik}{2ik}e^{-ika}\right)A$$

So we have

$$-\frac{\rho}{ik} \left(\frac{\rho + ik}{2ik}e^{ika} - \frac{\rho - ik}{2ik}e^{-ika}\right) = \left(\frac{\rho + ik}{2ik}e^{ika} + \frac{\rho - ik}{2ik}e^{-ika}\right)$$
Which means that
\[
\left( \frac{\rho - ik}{\rho + ik} \right)^2 = e^{2ika}
\]
So we have
\[
\exp[-4i \tan^{-1}\left( \frac{k}{\rho} \right)] = \exp[2ika]
\]
which means
\[
\tan^{-1}\left( \frac{k}{\rho} \right) = -\frac{1}{2} ka + \frac{1}{2} \pi n
\]
We have two solution for these equation:
First, \( \tan\left[ \frac{1}{2} ka \right] > 0 \) and
\[
\tan\left[ \frac{1}{2} ka \right] = \frac{\rho}{k}
\]
So
\[
\frac{1}{\cos^2(ka/2)} = 1 + \tan^2\left[ \frac{1}{2} ka \right] = \frac{k^2 + \rho^2}{k^2} = \left( \frac{k_0}{k} \right)^2
\]
where \( k_0 = \sqrt{\frac{2mV_0}{m}} \)
Second, \( \tan\left[ \frac{1}{2} ka \right] < 0 \) and
\[
\tan\left[ \frac{1}{2} ka \right] = -\frac{k}{\rho}
\]
So
\[
\sin^2(ka/2) = \frac{\tan^2\left[ \frac{1}{2} ka \right]}{1 + \tan^2\left[ \frac{1}{2} ka \right]} = \frac{k^2 + \rho^2}{k^2} = \left( \frac{k}{k_0} \right)^2
\]
Summarily we get the solution:
\[
\begin{cases}
|\cos\left[ \frac{1}{2} ka \right]| = \frac{k}{k_0} , \text{for} \tan\left[ \frac{1}{2} ka \right] > 0 \\
|\sin\left[ \frac{1}{2} ka \right]| = \frac{k}{k_0} , \text{for} \tan\left[ \frac{1}{2} ka \right] < 0
\end{cases}
\]
Now we solve this problem graphically. we set \( x = \frac{ka}{2} \), the straight line represents the function \( \frac{2}{k_0 a} x \), and sinusoidal arcs represent the functions \( |\cos\left[ \frac{1}{2} ka \right]| \) and \( |\cos\left[ \frac{1}{2} ka \right]| \), that satisfy the condition.
So it is easy to read from the graph that to have exact 2 solution we need:

\[
\frac{1}{\pi} < \frac{2}{k_{0}a} < \frac{2}{\pi}
\]

So:

\[
V_{1} \leq V_{0} \leq 4V_{1}
\]

where

\[
V_{1} = \frac{\pi^{2}h^{2}}{2ma^{2}} = 37.6eV
\]
Problem 2:

Consider a particle of mass $m$ confined in a one-dimensional infinite potential well

$$V = \begin{cases} 
0, & 0 < x < L \\
\infty, & \text{otherwise}
\end{cases}$$

Suppose that the particle is in the stationary state,

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

of energy $E_n = \frac{n^2\hbar^2}{2mL^2}$. Calculate

(i) $\langle x \rangle$ and $\langle p \rangle$

(ii) $\langle x^2 \rangle$ and $\langle p^2 \rangle$

(iii) $\Delta x \Delta p$

(4pts)

Solution

(i)

$$\langle x \rangle = \int_0^L \frac{2}{L} \sin\left(\frac{n\pi x}{L}\right)x \, dx$$

$$= \int_0^L \frac{1}{L}(1 - \cos\left(\frac{2n\pi x}{L}\right))x \, dx$$

$$= \frac{L}{2} + \int_0^L \frac{1}{2n\pi} \sin\left(\frac{2n\pi x}{L}\right) \, dx$$

$$= \frac{L}{2}$$

(0.1)

$$\langle p \rangle = 0$$

as the particle go nowhere.

(ii)

$$\langle x^2 \rangle = \int_0^L \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right)x^2 \, dx$$

$$= \int_0^L \frac{1}{L}(1 - \cos\left(\frac{2n\pi x}{L}\right))x^2 \, dx$$
\[ \frac{L^2}{3} + \int_0^L \frac{1}{n\pi} \sin\left(\frac{2n\pi x}{L}\right) \, dx \]
\[ = \frac{L^2}{3} - \frac{L}{2n^2\pi^2} \left. \cos\left(\frac{2n\pi x}{L}\right) \right|_0^L \]
\[ = \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2} \]

From \( E_n = \frac{\pi^2\hbar^2 n^2}{2mL^2} \) we get

\[ \langle p^2 \rangle = \frac{\pi^2\hbar^2 n^2}{L^2} \]

(iii)

\[ \Delta x = \left( \langle x^2 \rangle - \langle x \rangle^2 \right)^{\frac{1}{2}} = \left( \frac{L^2}{12} - \frac{L^2}{2n^2\pi^2} \right)^{\frac{1}{2}} \]

\[ \Delta p = \left( \langle p^2 \rangle - \langle p \rangle^2 \right)^{\frac{1}{2}} = \frac{\pi\hbar n}{L} \]

So we have

\[ \Delta x \Delta p = n\pi\hbar \sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}} \]
**Problem 3**
Consider a Hermitian operator $A$ that has the property $A^3 = 1$, Show that $A = 1$.

*Hint*: As the operator is Hermitian, the eigenvalue is real. (4pts)

**Solution**

First, we find the possible eigenvalues of $A$. Suppose $A|\Psi\rangle = \alpha|\Psi\rangle$, so we have

$$A^3|\Psi\rangle = \alpha^3|\Psi\rangle$$

As $A^3 = 1$, we have $\alpha^3 = 1$, which imply

$$\alpha = 1, e^{2\pi i/3}, e^{4\pi i/3}$$

Now as the operator is hermitian, $\alpha$ is real. So we have $\alpha = 1$. as the only possible eigenvalue is 1, we have $A = 1$. 
**Problem 4:**

Compute the matrix elements of the operators $x$ and $p$ for the one-dimensional harmonic oscillator.

\[ x_{nk} = \langle n|x|k \rangle = \int_{-\infty}^{\infty} \phi_n^*(x)x\phi_k(x)dx \]

\[ P_{nk} = \langle n|p|k \rangle = \int_{-\infty}^{\infty} \phi_n^*(x)p\phi_k(x)dx \]

Where $\phi_k(x)$ are the eigenfunctions of the harmonic oscillator.

*Hint:* write $x$ and $p$ in terms of lowering and raising operators.(4pts)

**Solution**

As we know

\[ x = \sqrt{\frac{\hbar}{2mw}}(a + a^\dagger), \quad p = i\sqrt{\frac{m\hbar}{2}}(a^\dagger - a) \]

On the other hand:

\[ a^\dagger|k\rangle = \sqrt{k + 1}|k + 1\rangle, \quad a|k\rangle = \sqrt{k}|k - 1\rangle \]

So we have

\[ x_{nk} = \langle n|x|k \rangle = \sqrt{\frac{\hbar}{2mw}}\langle n|a + a^\dagger|k \rangle = \sqrt{\frac{\hbar}{2mw}}\left( \langle n|\sqrt{k}|k - 1\rangle + \langle n|\sqrt{k + 1}|k + 1\rangle \right) \]

So we have:

\[ x_{nk} = \sqrt{\frac{\hbar k}{2mw}}\delta_{n,k-1} + \sqrt{\frac{\hbar(k + 1)}{2mw}}\delta_{n,k+1} \]

Similarly,

\[ p_{nk} = \langle n|p|k \rangle = i\sqrt{\frac{m\hbar}{2}}\langle n|a^\dagger - a|k \rangle = i\sqrt{\frac{m\hbar}{2}}\left( \langle n|\sqrt{k + 1}|k + 1\rangle - \langle n|\sqrt{k}|k - 1\rangle \right) \]

So we have:

\[ p_{nk} = i\sqrt{\frac{m\hbar(k + 1)}{2}}\delta_{n,k+1} - i\sqrt{\frac{m\hbar k}{2}}\delta_{n,k-1} \]
Problem 5:
Assume that in the Schrödinger picture all the operators are time-independent.

(i) Work in the Heisenberg picture and drive an equation expressing the
time evolution of an operator $A_H(t)$

(ii) Show that equation:
$$\frac{d\langle A \rangle}{dt} = \frac{1}{i\hbar} \langle [A, H(t)] \rangle$$
is also valid in the Heisenberg picture.

(4pts)

SOLUTION

(i) As we have assumed in the Schrödinger picture all the operators are
time-independent, we have
$$U(t, t_0) = e^{-iH(t-t_0)/\hbar}$$

So we have
$$A_H(t) = U^\dagger(t, t_0)AU(t, t_0) = e^{iH(t-t_0)/\hbar} A e^{-iH(t-t_0)/\hbar}$$

So we have
$$\frac{dA_H(t)}{dt} = iHA_H(t)/\hbar - A_H(t)iH/\hbar = \frac{i}{\hbar} [H, A_H(t)]$$

(ii) In the Heisenberg picture we have
$$\langle A(t) \rangle = \langle \Psi | A_H(t) | \Psi \rangle$$

On the right hand side of the above equation only $A_H(t)$ depends on time.

So we have
$$\frac{d\langle A(t) \rangle}{dt} = \langle \Psi | \frac{dA(t)}{dt} | \Psi \rangle$$

which means
$$\frac{d\langle A(t) \rangle}{dt} = \frac{i}{\hbar} \langle [H, A_H(t)] \rangle$$