LAST TIME:
Started chapter 22 on alternating currents

Discussed 3 types of circuit and derived the voltage drop for each element:

- Resistor \( R \):
  \[ V_R = IR \quad \phi = 0 \]

- Inductor \( X_L = \omega L \):
  \[ V_L = IX_L \quad \phi = +\frac{\pi}{2} \]

- Capacitor \( X_C = \frac{1}{\omega C} \):
  \[ V_C = IX_C \quad \phi = -\frac{\pi}{2} \]
Phasor diagram:
\[ i(t) = I \cos(\omega t) \; ; \; v(t) = V \cos(\omega t + \phi) \]

SERIES RLC CIRCUIT

The goal is to describe a circuit of the form:

![RLC Circuit Diagram]
Kirchhoff’s loop rule: the instantaneous voltage drop $v(t)$ coming from all 3 elements equal the voltage drop provided by the source at that moment.

Also notice that since the 3 elements are in series, then the current $i$ is the same for all 3 elements:

$$i(t) = I \cos(\omega t)$$

**Goal:** For the total instantaneous voltage drop

$$v(t) = V \cos(\omega t + \phi)$$

compute $V, \phi$ in terms of $R, C, L, \omega$. 
It is easiest to do the calculation by drawing the phasor diagram:

Case 1: \( X_L > X_C \)

By the Pythagorean theorem, the magnitude of \( \vec{V} \) is

\[
V = \sqrt{V_R^2 + (V_L - V_C)^2} = \sqrt{(IR)^2 + I(X_L - X_C)^2} = IZ
\]
A quantity satisfying
\[ V = I \cdot Z \]
Def. of impedance
\[ \uparrow \quad \uparrow \quad \text{total current amplitude} \]
\[ \text{total voltage amplitude} \]
is called impedance. This def. is general (also for systems in parallel!)

For the RLC circuit in series the impedance follows from our calculation.

\[ Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} \]

\[ X_L - X_C \equiv X = "\text{Reactance}" \]
From the phasor diagram we see:

\[ \tan \phi = \frac{V_L - V_c}{V_R} = \frac{I_0 (X_L - X_c)}{I R} = \frac{X_L - X_c}{R} \]

\[ \Rightarrow \phi = \arctan \left( \frac{wL - \frac{1}{wC}}{R} \right) \]

For \( X_L > X_c \) we see \( \phi > 0 \) so that \( 0 \leq \phi \leq \frac{\pi}{2} \).

**Case 2: \( X_L < X_c \)**

The phasor diagram is now:
The formulas previously derived for the voltage amplitude and the phase are still valid just with $\phi < 0$ so that $0 \leq \phi \leq -\frac{\pi}{2}$ as we see from the phasor diagram.

**NOTE:** Formulas are also valid for a circuit with less elements. by setting $R = 0$, $L = 0$ or $C = \infty$
EXERCISE

Limiting behavior of impedance

The figure shows an RCL circuit constructed by a student in a lab class. The resistance is due to the filament in a light bulb. When the student closes the switch, the bulb won't light up. Possible explanations are:
A. Inductance is too small
B. Frequency is too small ✓
C. Capacitance is too large
d. Frequency is too large ✓

**Solution:**
The bulb won't light up if the current is too small!
The amplitude of the current is

\[ I = \frac{V}{Z} \text{ with } Z = \sqrt{R^2 + \left(\frac{1}{L} - \frac{1}{\omega C}\right)^2} \]

So if \( Z \) is too large, \( I \) is too small!
We can get a large \( Z \) with a very large \( L \) or a very small \( \omega \)!

So B, D is the answer!
EXAMPLE: An RLC circuit

\[
\begin{align*}
R &= 300 \text{ Ohm} \\
L &= 60 \text{ mH} \\
C &= 0.50 \mu\text{F} \\
V &= 50 \text{ V}
\end{align*}
\]

\[\omega = 10.000 \frac{\text{rad}}{\text{s}}\]

Find: reactances \(X_L, X_C\), impedance \(Z\),
reactance \(X\), current amplitude \(I\),
phase angle \(\phi\), voltage amplitude
across each element.

Solution: By definition

\[X_L = \omega L = 10.000 \frac{\text{rad}}{\text{s}} \cdot 6 \cdot 10^{-3} \text{ H} = 600 \Omega\]
$X_C = \frac{1}{\omega C} = \frac{1}{10,000 \text{ rad/s} \cdot 0.50 \cdot 10^{-6} \text{ F}} = 200 \Omega$

The reactance then follows:

$X = X_L - X_R = 600 \Omega - 200 \Omega = 400 \Omega$

The impedance $Z$ is

$Z = \sqrt{R^2 + X^2} = \sqrt{(300 \Omega)^2 + (400 \Omega)^2} = 500 \Omega$

With the voltage amplitude $V = 50 \text{ V}$, the current amplitude is

$I = \frac{V}{Z} = \frac{50 \text{ V}}{500 \Omega} = 0.10 \text{ A}$
The phase angle $\phi$ is

$$\phi = \arctan \frac{X_L - X_C}{R} = \arctan \frac{400\Omega}{300\Omega} = 53^\circ$$

Because the phase is positive, the voltage leads the current by $53^\circ$

The voltage amplitude across the resistor follows from Ohm's law

$$V_R = IR = 0.10A \cdot 300\Omega = 30V$$

$$V_L = IX_L = 0.10A \cdot 600\Omega = 60V$$

$$V_C = IX_C = 0.10A \cdot 200\Omega = 20V$$
RMS Values:

The relation

\[ V = IZ \]

was written in terms of amplitudes. It is, of course, easy to rewrite such a relation in terms of rms values

\[ \frac{V}{\sqrt{2}} = \frac{I}{\sqrt{2}} Z \Rightarrow \]

\[ V_{\text{rms}} = I_{\text{rms}} Z \]