LAST TIME

* Started discussion about magnetic fields and magnetic forces
* We saw that when a charged particle travels through a magnetic field it experiences a force

\[ \vec{F}_B = q \cdot \vec{V} \times \vec{B} = q \cdot \vec{V} \sin \phi \cdot \vec{B} \]

\[ \vec{B} \uparrow \phi \uparrow \vec{V} \]

charge \hspace{1cm} velocity
Described the motion of charged particles in a $\mathbf{B}$-field

\[ R = \frac{mV}{lqH} \]

Cyclotron

Described the generalization to currents in a $\mathbf{B}$-field and current carrying conductors.

\[ \mathbf{F_B} = I \cdot l \mathbf{B} \sin \phi \]

\[ \mathbf{\hat{B}} \]
GOAL of today:

generalize the force acting on a straight wire to the force/torque of a loop.

**Force/torque of a current loop**

\[ |F_B| = BI \alpha \]
\[ |F_B'| = BI b \cdot \sin(90 - \phi) \]

\( \vec{m} \): normal to loop. Direction from right hand rule.
Please look at the figure in Young-Sellers to get all angles and orientations right.

We would like to show:

* The net force on the conductor vanishes
* There is a non-vanishing torque with interesting properties

**FORCE:**

* Force on Segment ①:
  \[ \vec{B} \perp \vec{I} \; j \; F_B = IaB \]
* Force on Segment ③:
  \[ \vec{B} \perp \vec{I} \; j \; F_B' = -F_B = -IaB \]

Thus ① and ③ cancel!
* Force on segment ②:
The angle between $\mathbf{B}$ and $\mathbf{i}$ is $90^\circ - \phi$

\[ F_B = IBb \sin(90^\circ - \phi) = IBb \cos \phi \]

* Force on segment ④

\[ F_B' = -F_B = -IBb \cos \phi \]

Thus ② and ④ cancel!

The total force on the loop is zero because the forces on opposite sides cancel pairwise!
**Torque**: There is however a torque; we can roughly think of torque as an "angular force" which causes a change in a rotational motion.

\[ \mathbf{\tau} = \mathbf{r} \times \mathbf{F} \]

\[ [\tau] = N \cdot m \]

\( \mathbf{F}' \) and \( -\mathbf{F}' \) are on a line so that \( \mathbf{\tau} \) vanishes. \( \mathbf{F}' \) and \( -\mathbf{F}' \) are equal in magnitude, opposite in direction and do not act on a line. These forces form a "couple" or "moment."
The torque of a couple is the product of the force times the perpendicular distance between the lines of action of the two forces.

From Fig. 20.27 of Young-Geller we see that this distance is:

\[ r = b \sin \phi \]

Thus the torque is:

\[ \tau = \frac{IaB \cdot b \sin \phi}{F} = I \cdot \frac{A}{r} \cdot B \sin \phi \]

\[ = \mu \cdot B \sin \phi \]

with \[ \mu = I \cdot A \]

Magnetic Dipole Moment.

This definition of magnetic moment.
as the product of the surface of the loop times the current going through the loop, holds for a loop of any geometry, not only rectangular loops!!

\[ \tau = \mu B \sin \phi \]

\( \phi \) = angle between the normal of the loop and the \( B \)-field.

* The torque is maximal for \( \phi = 90^\circ \)

\( \Phi = 90^\circ \)

\( B \) is in the plane of the loop
The torque vanishes for $\phi = 0$, $\phi = \pi$

$\vec{B}$ is perpendicular to the loop

$\phi = 0$ = stable equilibrium (if you computed the potential energy it would have a minimum)

The torque is zero for $\phi = 0$ and when the loop is slightly rotated to $\phi \neq 0$ it rotates back to $\phi = 0$

$\phi = 180^\circ$ is an unstable equilibrium (maximizes the energy).
\( \phi = 0 \) stable

\( \phi = 180^\circ \) unstable

\[ \mathcal{L} = \mathcal{M}B \sin \phi \]
holds for a loop of any shape

loops of any shape are made of rectangles.

**Generalization to solenoid**

Multiply the previous result by \( N \), number of windings!

\[ \mathcal{L} = (N)IAB \sin \phi \]

area of 1 loop
Exercise: stability of a current loop

A current loop is placed in a permanent magnet, as shown in the figure.

A. This is a stable equilibrium configuration.
B. This is an equilibrium configuration but not a stable one.
C. This is not an equilibrium.

Which one of these statements is correct?

Answer: B
The angle between $\mathbf{B}$ and $\mathbf{n}$ is 180° so that the equilibrium is unstable.

**NEXT:**
Magnetic field of a long straight conductor

Until now we have considered the motion of charged particles (single particles, straight wires or loops) in an **external** magnetic field, but we did
not worry about the origin of this field. (it could be a permanent magnet or an electromagnet).

**Goal**: Understand how such magnetic fields are produced from the motion of currents.

We shall consider different wire configurations of increasing difficulty.

**Magnetic Field of Long Straight Conductor**

- $\mathbf{B}$: radial field
- Magnetic field lines
\[ B \text{ has a constant magnitude on each field line:} \]
\[ B = \frac{\mu_0 I}{2\pi r} \]

Long straight conductor

\[ \mu_0 = \text{permeability of vacuum} \]

\[ [\mu_0] = \frac{T \cdot m}{A} \]
\[ \mu_0 = 4\pi \cdot 10^{-7} \ T \cdot m \over A \]

Remark: do not confuse \( \mu_0 \) with \( \mu \). The magnetic moment. Both quantities are completely unrelated!

Direction of \( B \) field: follows from the right hand rule: Grasp the conductor
with your right hand, your thumb pointing in the direction of the current. The direction in which your fingers curl is the direction of the B-field.

Once we have the B-field of 1 wire, we can compute the force between 2 wires...

**FORCE BETWEEN PARALLEL CONDUCTORS**

Two conductors are given: one conductor generates a magnetic field that the second conductor feels.
We can compute the force between both conductors.

The $B$-field that conductor 1 creates at the position of conductor 2 is:

$$B = \frac{\mu_0 I^{(1)}}{2\pi r}$$

$r$ = distance between both conductors

($F_B$ follows from right hand rule)
The force on a length \( l \) of the conductor \( \textcircled{2} \) is:

\[
F_B = I^{(2)}l \, B^{(1)} = I^{(2)}l \cdot \left( \frac{\mu_0 \cdot I^{(1)}}{2\pi r} \right)
\]

\[
F_B = \mu_0 \cdot l \frac{I^{(1)}I^{(2)}}{2\pi r}
\]

The force that the conductor \( \textcircled{2} \) generates on \( \textcircled{1} \) is \(-F_B\). Both wires attract each other!!
EXERCISE

Two straight parallel superconducting cables 4.5 mm apart (between centers) carry equal currents of 15.000 A in opposite directions. Find the magnitude and direction of the force per unit length exerted by one conductor on the other.
Solution:

To find the force use

\[ \frac{F}{\ell} = \mu_0 \frac{I I'}{2\pi r} \]

\[ \frac{F}{\ell} = \frac{4\pi \cdot 10^{-7} \frac{N}{A^2}}{2\pi} \left( 15.000 \, \text{A} \right)^2 \]

\[ \frac{F}{\ell} = \frac{4\pi \cdot 10^{-7} \frac{N}{A^2}}{2\pi} \left( 4.5 \cdot 10^{-3} \, \text{m} \right) = 1.0 \cdot 10^4 \frac{N}{m} \]

MORE COMPLICATED WIRE GEOMETRIES

There are some more complicated wire geometries for which the magnetic field is known. We will later on compute some of these $B$-fields.
using the law of Biot-Savart and Ampere's law. For now we will quote the experimental result

**Magnetic field at center of current loop**

![Diagram showing magnetic field at center of loop]

The direction of the field follows from right-hand rule.

Magnetic field at center of loop:

\[
B = \frac{\mu_0 I}{2R}
\]
**Magnetic Field at center of a coil**

If we instead have a coil with $N$ turns, then the magnetic field at the center is the previous result for one turn multiplied by the number of turns.

\[ B = \frac{\mu_0 N I}{2\pi R} \]

**B-field of a Solenoid**

A solenoid is a helix of wire wound around a cylinder.
Ideal solenoid: is very long \( L \gg R \) and its windings are closely spaced

* \( \vec{B} \)-field is homogenous inside (independent of position)
* \( \vec{B} \)-field vanishes outside

\[
m = \frac{N}{L}
\]

\[
\vec{B} = \mu_0 n I
\]  
\( \vec{B} \)-field inside ideal solenoid
*Magnetic Field inside a toroid*

A toroid is a solenoid in the form of a bagel.

If the individual windings are closely spaced, then the $B$-field is only non-vanishing inside the toroid:

$$B = \frac{\mu_0 NI}{2\pi r}$$
Notice that the field is not homogeneous but has a \( \frac{1}{r} \) dependence (where \( r \) is the distance to the center).

If \( r \gg R \) then the \( \vec{B} \)-field is nearly homogeneous and we recover a straight solenoid

\[
\vec{B} = \mu_0 \frac{N \cdot I}{2\pi r} \rightarrow \mu_0 \frac{N \cdot I}{L}
\]

\( L = \text{length of circle} \)
EXERCISE:
A bar magnet is suspended freely as shown in the figure.

If the magnet aligns so that the N pole faces the solenoid in which orientation is the battery connected, (A) or (B)?

ANSWER: (A) use right hand rule