Rotational Motion

I. Rotational Kinematics
II. Rotational Dynamics
   (Newton’s Law for Rotation)
III. Angular Momentum Conservation

Part I

1. Remember how Newton’s Laws for translational motion were studied:
   1. Kinematics ($x = x_0 + v_0 t + \frac{1}{2} a t^2$)
   2. Dynamics ($F = m a$)
   3. Momentum Conservation
2. Now, we repeat them again, but for rotational motion:
   1. Kinematics ($\theta, \omega, \alpha$)
   2. Dynamics ($\tau = I \alpha$)
   3. Angular Momentum
Newton’s Laws for Rotation

\[ \vec{\tau}_{\text{net}} = I \ddot{\alpha} \]

1st part

2nd part

3rd part

Kinematical variables to describe the rotational motion:

Angular position, velocity and acceleration

\[ \theta = \frac{l}{R} \quad \text{(rad)} \]

\[ \omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt} \quad \text{(rad/s)} \]

\[ \alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt} \quad \text{(rad/s}^2) \]
### Linear and Angular Quantities

Angular position, velocity and acceleration:
- c.w. or c.c.w. rotation (like +x or –x direction in 1D)

### Vector Nature of Angular Quantities

**Kinematical variables to describe the rotational motion:**
- Angular position, velocity and acceleration
- R.-H. Rule

\[
\theta = \frac{l}{R} \quad \text{(rad)}
\]
\[
\omega \hat{k} = \frac{d\theta}{dt} \hat{k} \quad \text{(rad/s)}
\]
\[
\alpha \hat{k} = \frac{d\omega}{dt} \hat{k} \quad \text{(rad/s^2)}
\]

≥0 or <0
Example 1 (α = constant)

(1) What is the angular speed of rotation of the Earth?
   1 rotation (2π rad) per 24 hours (1440 min)

(2) An old 33 1/3 rpm record player starts from rest and reaches operating speed in 2.00 seconds. Through what angle did it turn in those 2.00 seconds?
   $t = 2.00 \text{ s}, \omega_0 = 0, \omega = 33 \frac{1}{3} \text{ rpm}$

(3) A computer hard drive rotates at 5400 rpm. What angular acceleration will get it up to speed in just 150 revolutions starting from rest?
   Use Eq. (3) with $\theta - \theta_0 = 150 \text{ rev.} = ? \text{ rad}$

Example 1 (cont’d)

(4) A hard drive reaches 5400 rpm in 3.20 seconds. What was the average angular speed assuming constant angular acceleration?

(5) A dentist’s drill accelerates to 1800 rpm in 2.50 seconds. What is its angular acceleration?

(6) The angular velocity changes from 47.0 rad/s to −47.0 rad/s in 2.00 seconds. What is the angular acceleration?
Example 2

A grinding wheel turns at a constant angular acceleration of 60.0 rad/s² from 24.0 rad/s for 2.00 sec. Then, a circuit breaker trips. It turns through 432 rad as it coasts to a stop at a constant angular acceleration.

Find:
(a) the total angle between \( t = 0 \) and the time it stopped;
(b) the time it stopped;
(c) the angular acceleration as it slowed down.

Also sketch  \( \theta \) vs.  \( t \) graph.

Practice Problem 1

Problem 3: (25 points)
A grinding wheel turns at a varying angular acceleration of \( \alpha(t) = (30.0 \text{ rad/s}^2) t \) for 2.00 sec. Assume the initial angular speed of 20.0 rad/s. Then, a circuit breaker trips. It turns through 400 rad as it coasts to a stop at a constant angular acceleration.

a. (5 pts) Find the total angle (\( \theta_{\text{total}} \)) between \( t = 0 \) and the time it stopped.
b. (10 pts) Find the time (\( t_{\text{total}} \)) it stopped. Find the angular acceleration as it slowed down.
c. (10 pts) Sketch \( \theta \)-\( t \), \( \omega \)-\( t \), \( \alpha \)-\( t \) graphs.
**Practice Problem 2**

A solid cylinder (radius \( R = 2 \) m and height \( H = 5 \) m) turns at a constant angular acceleration of \( 60.0 \text{ rad/s}^2 \) from \( 24.0 \) rad/s for \( 2.00 \) sec. Then, a circuit breaker trips. It turns through \( 432 \) rad as it coasts to a stop at a constant angular acceleration.

(a) Find the total angle between \( t = 0 \) and the time it stopped.
(b) Find the time it stopped.
(c) Find the angular acceleration as it slowed down.
(d) Find the speed \((v)\) of point \( P\) at \( t = 2.00 \) sec.
(e) Sketch the motion of point \( P \) in \( v-t \) graph. Also sketch \( \theta t \) graph.

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**Part II**

1. Remember how Newton’s Laws for translational motion were studied:
   1. Kinematics \( x = x_0 + v_0 t + \frac{1}{2} a t^2 \)
   2. Dynamics \( F = m a \)
   3. Momentum Conservation

2. Now, we repeat them again, but for rotational motion:
   1. Kinematics \( \theta, \omega, \alpha \)
   2. Dynamics \( \tau = I \alpha \)
   3. Angular Momentum
Torque due to Gravity?

\[ \tau = \vec{r} \times \vec{F} = ? \]

We assume:

Center of mass (CM or cm) = Center of gravity (if uniform gravity)

Rotational Motion
Net Torque due to Gravity?

One extra sphere!
Do we need a new concept? No!

\[ \vec{\tau}_{\text{net}} = \vec{\tau}_{\text{rod}} + \vec{\tau}_{\text{sphere}} \]

Solid sphere \((M, r)\)

Note: sign of \(\tau\) and \(\Sigma \tau\)

\[ \tau_1 = F_1 \left( R_1 \sin 90^\circ \right) \]
\[ = (50.0 \text{N})(0.300 \text{ m}) = 15.0 \text{ N} \cdot \text{m} \]

\[ \tau_2 = F_2 \left( R_2 \sin 60^\circ \right) \]
\[ = (50.0 \text{N})(0.500 \text{ m})(0.866) \]
\[ = 21.7 \text{ N} \cdot \text{m} \]

\[ \vec{\tau}_{\text{net}} = \vec{\tau}_1[\text{c.c.w.}] + \vec{\tau}_2[\text{c.c.w.}] \]
\[ = \tau_1[1] + \tau_2[-1] \]
\[ = [(15.0 \text{N} \cdot \text{m}) - (21.7 \text{N} \cdot \text{m})][\text{c.c.w.}] \]
\[ = -6.7 \text{ N} \cdot \text{m}[\text{c.c.w.}] \rightarrow 6.7 \text{ N} \cdot \text{m}[\text{c.w.}] \]
Example 1

Calculate the torque on the 2.00-m long beam due to a 50.0 N force (top) about
(a) point C (= c.m.)
(b) point P

Calculate the torque on the 2.00-m long beam due to a 60.0 N force about
(a) point C (= c.m.)
(b) point P

Calculate the torque on the 2.00-m long beam due to a 50.0 N force (bottom) about
(a) point C (= c.m.)
(b) point P

Example 1 (cont’d)

Calculate the net torque on the 2.00-m long beam about
(a) point C (= c.m.)
(b) point P
Practice Problem 2

Determine the net torque (magnitude and direction) on the 2.00-m-long beam about:

a. (15 pts) point C (the CM position);
b. (10 pts) point P at one end.

\[ \vec{\tau}_i = \vec{r}_i \times \vec{F}_i = ? \]

\[ \vec{\tau}_{\text{net}} = \sum_i \vec{\tau}_i \]

Practice Problem 2

Determine the net torque (magnitude and direction) on the disk about pin.

\[ \vec{\tau}_i = \vec{r}_i \times \vec{F}_i = ? \]

\[ \vec{\tau}_{\text{net}} = \sum_i \vec{\tau}_i \]

\[ |\vec{F}_1| = 50 \text{ N} \]

\[ |\vec{F}_2| = 25 \text{ N} \]
Newton’s Laws for Rotation

\[ \bar{\tau}_{\text{net}} = I \bar{\alpha} \]

- 2nd part [N m]
- 1st part [s^{-2}]
- 3rd part

Parallel-axis Theorem

\[ I = I_1 + I_2 \]

Rotation Motion
2(a) Express the moment of inertia of the array of point objects about the $y$-axis in terms of $m$, $M$, $X_1$, $X_2$, and/or $Y$.

![Diagram of point objects and axes]

Newton’s Laws for Rotation

$$\vec{\tau}_{\text{net}} = I \vec{\alpha}$$

2nd part [N m]  
3rd part [kg m$^2$]  
1st part [s$^{-2}$]

$K_{rot} = (1/2) I \omega^2$ ( $K = (1/2) m v^2$)

Application?
Rolling Motion (w/o slipping)

Rotational Motion

Rolling Motion w/o Slipping (2)

Translation

\[ \omega \]

Rotation

\[ \omega \]

Rolling

\[ \omega \]

Instantaneously rest
Instantaneous axis

Faster

Rotational Motion
Example 3

What will be the speed of each of the following objects when it reaches the bottom of an incline if it starts from rest at a vertical height $H$ and rolls without slipping? Find the speed using the conservation of mechanical energy.

(i) Solid sphere $(M, R_o)$
(ii) Solid cylinder $(M, R_o, l)$
(iii) Thin hoop $(M, R_o, l)$

Example 4

Analyze the rolling sphere in terms of forces and torques: find the magnitudes of the velocity $v$ and the friction force $F_{fr}$.

1. F.B.D.?

2. Is $F_{fr}$ static or kinetic friction?
   - Static Friction
   - No work
   - No energy loss
   - $K+U=\text{constant}$

Thus:
- The motion of the wheel is a pure rotation about the instantaneous axis through $P$ at the instant.

Rotational Motion
**Example 5**

A string is wrapped around a uniform solid cylinder of mass \( M (= 0.500 \text{ kg}) \) and radius \( R (= 0.200 \text{ m}) \), and the cylinder starts falling from rest. Draw the free-body diagram for the cylinder while it descends. Also find (a) its acceleration, (b) the tension in the string, and (c) the speed after the cylinder has descended \( h = 0.800 \text{ m} \).

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**Example 5 - Challenge**

A string is wrapped around a hollow cylinder of mass \( M (= 0.500 \text{ kg}) \), inner radius \( R_1 (= 0.100 \text{ m}) \), outer radius \( R_2 (= 0.200 \text{ m}) \). The cylinder starts falling from rest. (a) Draw the free-body diagram for the cylinder while it descends. Also find:
(b) its acceleration;
(c) the tension in the string;
(d) the speed after the cylinder has descended \( h = 0.800 \text{ m} \).