PHYSICS 218 Final Exam

December 12, 2003

Name (Last, First): __________________________

Section: ____________ SID (last 5 digits): __-__-__-__-

• You have 120 minutes to complete the exam.
• Formulae are provided on a separate colored sheet. You may NOT use any other formula sheet.
• You may use a calculator.
• When calculating numerical values, be sure to keep track of units. Results must include proper units.
• Be alert to the number of significant figures in the information given. Results must have the correct number of significant figures.
• If you are unable to solve a problem whose solution is needed in another problem, then assign a symbol for the solution of the first problem and use that symbol in solving the second problem.
• If you need additional space to answer a problem, use the back of the sheet it is written on.
• Mark your answers clearly by drawing boxes around them.
• Please write clearly. You may gain marks for a partially correct calculation if your work can be deciphered.
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Problem 1: (50 points) – Newton’s 2nd law of the rolling motion

Consider a thin, hollow pipe rolling down an incline. Assume the pipe has mass $M$, radius $R_0$. The pipe is placed at rest at the top of an incline a height $H$ above the ground. The incline is at an angle $\theta$ as shown in the figure. It then rolls down the incline without slipping. Note that the magnitude of the gravitational acceleration near the Earth’s surface is $g$. Ignore losses due to dissipative forces. Also note friction ($F_f$) is static so that we cannot assume $F_f = \mu_s F_N$. Only $F_f \leq \mu_s F_N$.

(a) (10 pts) Draw the free-body diagram for the pipe.

(b) (10 pts) Express the linear acceleration of the pipe in terms of $M$, $H$, $R_0$, $\theta$, and/or $g$.

(c) (10 pts) Using this acceleration, express the final speed of the pipe in terms of $M$, $H$, $R_0$, $\theta$, and/or $g$ when it reaches the ground.

(d) (10 pts) As a check, find the final speed using conservation of energy.

(e) (10 pts) If you use a solid sphere with $M$ and $R_0$, do you expect that the linear acceleration of the sphere increase, decrease or remain in the same? How about the final speed? Explain.
Problem 2: (50 points) – Momentum conservation in 2D

A rocket of mass $M$ in outer space, traveling with speed $v_0$ along the $x$ axis, suddenly shoots out one-third its mass that is moving off at angle $\theta = 60^\circ$ from the $x$ axis with speed $2v_0$. Ignore the influence of gravity.

(a) (10 pts) Sketch a diagram describing the motion.

(b) (20 pts) Express the velocity of the remainder of the object in terms of unit vectors $\mathbf{i}$ and $\mathbf{j}$.

(c) (20 pts) Is this “separation” elastic or inelastic? Why?
Problem 3: (50 points) – Angular momentum conservation

A person of mass $M = 55.0 \text{ kg}$ stands at the center of a rotating merry-go-round platform of radius $R = 2.50 \text{ m}$ and moment of inertia $I_{\text{platform}} = 670 \text{ kg} \cdot \text{m}^2$. The platform rotates without friction with an angular speed of $\omega_0 = 2.00 \text{ rad/s}$. The surface of the platform is rough, so that the person can walk radially without slipping. Treat the person as a particle.

(a) (10 pts) Express the moment of inertia of the system of platform plus person as a function of the person’s radial distance ($x$) from the center of the platform.

(b) (10 pts) Use the result in part (a) to find the numerical value of the moment of inertia of the system when the person stands at the center of the platform. Also find it when the person reaches to the edge ($x = R$).

(c) (10 pts) Use the result in part (b) to find the magnitude of the angular velocity of the system when the person reaches to the edge.

(d) (10 pts) Find the rotational kinetic energies of the system of platform plus person before and after the person’s walk.

(e) (10 pts) Use results in part (d) to determine the work done by the person during the walking.
Problem 4: (50 points) – S.H.M.

Galileo Galilei wanted to use a meter stick (mass $M = 0.500$ kg and length $L = 10.0$ cm = 0.100 m) and a disk (mass $m = 2.00$ kg and radius $r = 5.00$ cm = 5.00 x $10^{-2}$ m) as a pendulum. He planned to drill a small hole through the meter stick and suspended it from a smooth pin attached to the wall. See the figure below. Note the figure is not to scale. The center of mass (c.m.) of the stick was displaced a small angle ($\theta_0 = 2.00^\circ$) from the vertical and released. Note that the magnitude of the gravitational acceleration near the Earth’s surface is $g_{\text{earth}} = 9.80$ m/s$^2$.

(a) (20 pts) Express the moment of inertia of the pendulum about the pin in terms of $M$, $L$, $x$, $m$ and/or $r$. [Hint: use Parallel-axis theorem.]. Also calculate the numerical value for $x = 5.00$ cm.

(b) (15 pts) Find the numerical value of the period of small oscillations for $x = 5.00$ cm.

(c) (10 pts) Find the numerical value of the angular speed of the stick when it reached the vertical position. Assume $x = 5.00$ cm.

(d) (5 pts) If the pendulum (same length and radius, same $x$, but with masses $6M$ and $6m$) would be placed on the Moon ($g_{\text{moon}} = g_{\text{earth}}/6$), what would be the period of the oscillations? Would Galileo predict that it increased, decreased, or remained in the same? Explain your answer.
Problem 1: (50 points) – Newton’s 2nd law of the rolling motion

(a) (10 pts) Draw F.B.D. here

(b) (10 pts) Testing your knowledge of Newton’s 2nd laws.
   Step 1 (9 pts): Write down Newton’s 2nd laws. 3 points for each equation.
   \[ \vec{F} = m\vec{a} \]
   \[ \vec{t} = I\vec{\alpha} \]
   \[ [x] Mg \sin \theta - F_x = Ma_x \cdots (1) \]
   \[ F_x \cdot R_0 = MR_0^2 \left( \frac{a_x}{R_0} \right) \Rightarrow F_x = Ma_x \cdots (3) \]
   \[ [y] F_N - Mg \cos \theta = 0 \cdots (2) \quad \text{Direction: } \otimes \]
   Note that 3 equations and 3 unknowns \((a_x, F_x, \text{ and } F_N)\)

Step 2 (1 pt): Solve for \(a_x\).
   \[ (3) \rightarrow (1) : Mg \sin \theta - Ma_x = Ma_x \quad \therefore Ma_x = \frac{1}{2} Mg \sin \theta \quad \therefore a_x = \frac{1}{2} g \sin \theta \cdots (4) \]

(c) (10 pts) Testing your knowledge of kinematics.
   Eq.(4) \(\rightarrow a_x\) is constant. So \(v_x^2 - v_{0x}^2 = 2a_x (x - x_0)\) \(\langle 7 \text{ pts} \rangle\)
   \[ \therefore v_x^2 - 0 = 2a_x \frac{H}{\sin \theta} \cdots (5) \]
   \[ \therefore v_x^2 = 2 \left( \frac{1}{2} g \sin \theta \right) \left( \frac{H}{\sin \theta} \right) = gH \quad \therefore v_x = \sqrt{gH} \quad \langle 1 \text{ pt} \rangle \]

(d) (10 pts) Testing your knowledge of conservation of mechanical energy in rolling motion
   \(K_i + U_i = K_f + U_f\), where \(U\) = gravitational potential energy
   \[ 0 + MgH = \left( \frac{1}{2} Mv_x^2 + \frac{1}{2} I_{cm} \omega^2 \right) + 0, \quad \text{where } I_{cm} = MR_0^2 \text{ and } \omega = \frac{v_x}{R_0} \]
   \[ K_f = K_{trans} + K_{rotation} \quad \langle 3 \text{ pts} \rangle \]
   \[ \therefore MgH = \frac{1}{2} Mv_x^2 + \frac{1}{2} MR_0^2 \left( \frac{v_x}{R_0} \right)^2 \quad \therefore MgH = \frac{1}{2} Mv_x^2 + \frac{1}{2} Mv_x^2 = Mv_x^2 \quad \therefore v_x = \sqrt{gH} \]

(e) (10 pts) Testing your knowledge of rolling motion of different rigid body.
   Acceleration \(\langle 5 \text{ pts} \rangle\): Use \(I_{cm} = \frac{2}{5} MR_0^2\) in Eq.(3).
   \[ \therefore a_x (\text{solid sphere}) = \frac{5}{7} g \sin \theta \quad \therefore a_x (\text{solid sphere}) > a_x (\text{thin hollow pipe}) \quad \therefore \text{increase} \]
   Speed \(\langle 5 \text{ pts} \rangle\): Look back at Eq.(5).
   \[ \therefore v_x = \sqrt{\frac{10}{7} gH} \quad \therefore v_x \text{ increases when } a_x \text{ increases.} \]
Problem 2: (20 points) – Momentum conservation in 2D

A rocket of mass $M$ in outer space, traveling with speed $v_0$ along the $x$ axis, suddenly shoots out one-third its mass that is moving off at angle $\theta = 60^\circ$ from the $x$ axis with speed $2v_0$. Ignore the influence of gravity.

(a) (10 pts) Sketch a diagram describing the motion.

(b) (20 pts) Express the velocity of the remainder of the object in terms of unit vectors $i$ and $j$.

(c) (20 pts) Is this “separation” elastic or inelastic? Why?

Solutions:

a. 

\[\begin{align*}
\text{b. Momentum conservation:} \\
Mv_0 &= (2/3)Mv_{2x} + (1/3)M(2v_0) \cos \theta \quad \text{..... (7 pts)} \\
0 &= (1/3)M(2v_0) \sin \theta + (2/3)Mv_{2y} \quad \text{..... (7 pts)}
\end{align*}\]

Then:

\[\begin{align*}
v_{2x} &= (3/2)v_0 \left[ 1 - (2/3) \cos \theta \right] = v_0 \\
v_{2y} &= -v_0 \sin \theta = -\left[\sqrt{3}/2\right]v_0 = -0.866v_0
\end{align*}\]

Therefore:

\[\mathbf{v}_2 = \left[v_0\right]i + \left[-v_0\sin \theta\right]j \quad \text{or any other equivalent answer} \quad \text{..... (6 pts)}\]

cf.) magnitude: \(v_2 = \sqrt{7/4} \ v_0 = 1.32 \ v_0\)

direction: \(\phi = \tan^{-1}(0.866/1.000) = 40.9^\circ\)

c. “Inelastic” (5 pts), because $\Delta K (=K_f - K_i)$ (10 pts) is nonzero (5 pts):

\[\begin{align*}
K_f - K_i &= [(1/2)(M/3)(v_1)^2 + (1/2)(2M/3)(v_2)^2] - (1/2)M(v_0)^2 \\
&= 0.75 \ M \ (v_0)^2 \quad \text{..... Mechanical energy is not conserved.}
\end{align*}\]
Problem 3: (50 points) – Angular momentum conservation

A person of mass \( M = 55.0 \text{ kg} \) stands at the center of a rotating merry-go-round platform of radius \( R = 2.50 \text{ m} \) and moment of inertia \( I_{\text{platform}} = 670 \text{ kg} \cdot \text{m}^2 \). The platform rotates without friction with an angular speed of \( \omega_0 = 2.00 \text{ rad/s} \). The surface of the platform is rough, so that the person can walk radially without slipping. Treat the person as a particle.

(a) (10 pts) Express the moment of inertia of the system of platform plus person as a function of the person’s radial distance (\( x \)) from the center of the platform.

(b) (10 pts) Use the result in part (a) to find the numerical value of the moment of inertia of the system when the person stands at the center of the platform. Also find it when the person reaches to the edge (\( x = R \)).

(c) (10 pts) Use the result in part (b) to find the magnitude of the angular velocity of the system when the person reaches to the edge.

(d) (10 pts) Find the rotational kinetic energies of the system of platform plus person before and after the person’s walk.

(e) (10 pts) Use results in part (d) to determine the work done by the person during the walking.

(a) Use \( I = I_{\text{platform}} + I_{\text{person}} \) (5 pts). \( \therefore I_{\text{platform+person}} = (670 \text{ kg} \cdot \text{m}^2) + (55.0 \text{ kg}) \times x^2 \) (5pts)

(b) Set \( x = 0 \). \( \therefore I_{\text{platform+person}} = (670 \text{ kg} \cdot \text{m}^2) + (55.0 \text{ kg})(0.00 \text{ m})^2 = 670 \text{ kg} \cdot \text{m}^2 \) (5pts)

Set \( x = R \). \( \therefore I_{\text{platform+person}} = (670 \text{ kg} \cdot \text{m}^2) + (55.0 \text{ kg})(2.50 \text{ m})^2 = 1014 \text{ kg} \cdot \text{m}^2 \) (5pts)

(c) Use angular momentum conservation \([I_{i} \omega_{i} = I_{f} \omega_{f}] \) (5pts).

\[ \therefore \omega_{f} = \frac{I_{\text{platform}} \omega_{0}}{I_{\text{platform+person}}} = \frac{(670 \text{ kg} \cdot \text{m}^2)(2.00 \text{ rad/s})}{1014 \text{ kg} \cdot \text{m}^2} = 1.32 \text{ rad/s} \] (2pts)

(d) Use \( K_{\text{rot}} = \frac{1}{2} I \omega^2 \) (6 pts).

\[ K_{\text{rot before}} = \frac{1}{2} I_{\text{platform}} \omega_{0}^2 = \frac{1}{2}(670 \text{ kg} \cdot \text{m}^2)(2.00 \text{ rad/s})^2 = 1340 \text{ J} \] (2 pts)

\[ K_{\text{rot after}} = \frac{1}{2} I_{\text{platform+person}} \omega_{f}^2 = \frac{1}{2}(1014 \text{ kg} \cdot \text{m}^2)(1.32 \text{ rad/s})^2 = 883 \text{ J} \] (2 pts)

(e) \( W = K_{\text{rot after}} - K_{\text{rot before}} \) (9 pts) = -457 \text{ J} (1 pt)

The work done by the person is negative!

This means the energy is \textit{transferred} from the system of the platform plus person.
Problem 4: (50 points) – S.H.M.

(a) The system has two rigid bodies.

\[ I_{\text{system}} = I_{\text{stick}} + I_{\text{disk}} = \left(\frac{1}{12} ML^2 + M\left(\frac{L}{2} - x\right)^2\right) + \frac{1}{2} m r^2 + m(L - x - r)^2 \]

\[ \Rightarrow I_{\text{system}} = \frac{1}{12} ML^2 + M\left(\frac{L}{2} - x\right)^2 + \frac{1}{2} m r^2 + m(L - x - r)^2 \]

Use \( I = I_{\text{cm}} + Md^2 \)

\[ \Rightarrow \text{Use } I = I_{\text{cm}} + Md^2 \] \( (3 \text{ pts}) \)

\[ \therefore I_{\text{system}} = 0.022916 \cdots = 0.0229 \text{ kg} \cdot \text{m}^2 \text{ (or} \ 229 \text{ kg} \cdot \text{cm}^2) \text{ for } x = 0.05 \text{ m} \] \( 1 \text{ pt} \)

(b) Use \( \omega_{\text{freq}} = \sqrt{\frac{M \text{ system} g d_{\text{cm}}}{I_{\text{system}}}} \) \( 2 \text{ pts} \),

where \( d_{\text{cm}} \) is the c.m. position of the system from the pin \( 1 \text{ pts} \);

\( M_{\text{system}} \) is the mass of the system of stick and disk \( 2 \text{ pts} \);

\( I_{\text{system}} \) is the moment of inertia of the system from part (a) \( 1 \text{ pts} \).

\[ \therefore x_{\text{cm}} = \frac{M \left(\frac{L}{2}\right) + m(L + r)}{M + m} \] \( 2 \text{ pts} \) \( \therefore d_{\text{cm}} = 13.0 \text{ cm} - 5.0 \text{ cm} = 8.0 \text{ cm} \) \( 1 \text{ pt} \)

\[ \therefore \omega_{\text{freq}} = \frac{\sqrt{\frac{(M + m) g d_{\text{cm}}}{I_{\text{system}}}}}{\omega_{\text{freq}}} \quad \therefore \omega_{\text{freq}} = 9.25 \text{ s}^{-1} \] \( 1 \text{ pt} \)

\[ \therefore T = \frac{2\pi}{\omega_{\text{freq}}} = \frac{2\pi}{9.25} = 0.679 \text{ s} \] \( 1 \text{ pt} \)

(c) Use \( K_{\text{rot}} + U = K_{\text{rot2}} + U_2 \) \( 4 \text{ pts} \).

\[ \therefore 0 + (M + m)g d_{\text{cm}}(1 - \cos \theta_0) = \frac{1}{2} I_{\text{system}} \omega^2 + 0 \] \( 5 \text{ pts} \)

\[ \therefore \omega^2 = \frac{2(M + m)g d_{\text{cm}}(1 - \cos \theta_0)}{I_{\text{system}}} = \frac{2(M + m)g d_{\text{cm}} \theta_0^2}{I_{\text{system}}} \]

\[ \therefore \omega = \sqrt{\frac{(M + m)g d_{\text{cm}} \theta_0}{I_{\text{system}}}} \theta_0 = \omega_{\text{freq}} \times \theta_0 = (9.25 \text{ s}^{-1}) \times \left(200^\circ \cdot \frac{2\pi}{180^\circ}\right) = 0.323 \text{ s}^{-1} \] \( 1 \text{ pt} \)

in radians

or Use \( \theta(t) = \theta_0 \cos(\omega_{\text{freq}} t) \) \( 4 \text{ pts} \).

\[ \therefore \omega = -\omega_{\text{freq}} \theta_0 \sin(\omega_{\text{freq}} t) \] \( 5 \text{ pts} \)

\[ \therefore \omega_{\text{max}} = \omega_{\text{freq}} \theta_0 \left(\text{at} \quad t = \frac{T}{4}\right) = 0.323 \text{ s}^{-1} \] \( 1 \text{ pt} \)

(d) Look back at \( \omega_{\text{freq}} = \sqrt{\frac{(M + m)g d_{\text{cm}}}{I_{\text{system}}}} \);

\( I_{\text{system}} \rightarrow 6 \times I_{\text{system}} \) from part (a) \( 1 \text{ pt} \),

\( d_{\text{cm}} \rightarrow d_{\text{cm}} \) (unchanged) because \( x_{\text{cm}} \) is unchanged from part (b) \( 1 \text{ pt} \),

\( M + m \rightarrow 6(M + m) \) \( 1 \text{ pt} \)

\( g \rightarrow \frac{g}{6} \) \( 1 \text{ pt} \)

\[ \therefore \omega \rightarrow \frac{\omega}{\sqrt{6}} \Rightarrow \text{Smaller } \omega \text{ on the Moon means longer } T (T_{\text{moon}} > T_{\text{earth}}). \]

\[ \therefore \text{"increase"} \] \( 1 \text{ pt} \)
Problem 4: (50 points) – Solution – from lecture note

Calculate the c.m. position of the system.

See the figures in the right.

The answer is 13.0 cm from the one top end of the stick.