You have 60 minutes to complete the exam.
- Formulae are provided on a separate colored sheet. You may NOT use any other formula sheet.
- You may use a calculator.
- When calculating numerical values, be sure to keep track of units. Results must include proper units.
- Be alert to the number of significant figures in the information given. Results must have the correct number of significant figures.
- If you are unable to solve a problem whose solution is needed in another problem, then assign a symbol for the solution of the first problem and use that symbol in solving the second problem.
- If you need additional space to answer a problem, use the back of the sheet it is written on.
- Mark your answers clearly by drawing boxes around them.
- Please write clearly. You may gain marks for a partially correct calculation if your work can be deciphered.
Problem 1: (25 points) – Energy conservation

A trampoline artist (mass \( M = 75.0 \) kg) jumps vertically upward from the top of a platform with a speed of \( v_0 = 5.00 \) m/s. He lands on the trampoline, \( h = 2.00 \) m below. If the trampoline behaves like an ideal spring of spring constant \( k = 5.00 \times 10^4 \) N/m, how far does he depress it? The magnitude of the acceleration due to the Earth’s gravity is \( g = 9.80 \) m/s\(^2\). [5 Bonus Points] Express the magnitude of the depression in terms of \( M, v_0, h, k, \) and/or \( g \).

Let "\( d \)" be the magnitude of the depression and use the work-energy theorem.

\[
\begin{align*}
W_{sp} &= U_e(0) - U_e(-d) = 0 - \frac{1}{2} k (-d)^2 = -\frac{1}{2} kd^2 \\
W_g &= U_g(h) - U_g(-d) = Mgh - Mg(-d) \\
W_{net} &= Mgh + Mg d - \frac{1}{2} kd^2 \\
0 &= Mgh + Mg d - \frac{1}{2} kd^2 = \frac{1}{2} Mv_f^2 - \frac{1}{2} Mv_0^2 \\
\frac{1}{2} kd^2 - Mg d - Mgh - \frac{1}{2} Mv_0^2 &= 0 \\
\Rightarrow d &= \frac{Mg \pm \sqrt{(Mg)^2 + 4 \left( \frac{1}{2} k \right) (Mgh + \frac{1}{2} Mv_0^2)}}{2 \left( \frac{1}{2} k \right)} \\
&= \frac{Mg}{k} \left( 1 \pm \sqrt{1 + \frac{2k}{Mg} \left( h + \frac{v_0^2}{2g} \right)} \right) \\
&= 0.325 \text{ m}
\end{align*}
\]
Problem 2: (25 points) - Momentum conservation

Two billiard balls of masses \( m_1 = 4M \) and \( m_2 = M \) move at right angles and meet at the origin of an \( xy \) coordinate system. One is moving upward along the \( y \) axis at 2.00 m/s, and the other is moving to the right along the \( x \) axis with speed 3.70 m/s. After the collision (assumed elastic), the second ball is moving along the positive \( y \) axis. (a) What is the final direction of the first ball? What are their two speeds? (b) If \( m_1 \) is heavier, explain how the final direction is changed.

Assume \( m_1 \) moves in 1st quadrant after the collision. Any quadrant is OK.

Momentum conservation

\[ m_2 v_2 = m_1 v_1' \cos \theta \]
\[ m_1 v_1 = m_1 v_1' \sin \theta + m_2 v_2' \]
\[ v_1', v_2', \text{ and } \theta \text{ are unknown.} \]

Need one more equation. Use "elastic" collision: \( K_i = K_f \)

\[ \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \]

\( m_1 = 4M \) and \( m_2 = M \). So,

\[ \begin{cases} v_2' = 4 v_1' \cos \theta & \text{(1)} \\ 4 v_1' = 4 v_1' \sin \theta + v_2' & \text{(2)} \\ 4 v_1'^2 + v_2'^2 = 4 v_1'^2 + v_2'^2 & \text{(3)} \end{cases} \]

\( 1 \to 4 v_1' \cos \theta = v_2' \)
\( 2 \to 4 v_1' \sin \theta = 4 v_1' - v_2' \)
\( 3 \to 16 v_1'^2 = 16 v_1'^2 + 4 v_2'^2 - 4 v_2'^2 \)
\( 4, 5 \to v_2'^2 + (4 v_1' - v_2')^2 = 16 v_1'^2 + 4 v_2'^2 - 4 v_2'^2 \)
\( 5 v_2'^2 - 8 v_1' v_2' - 3 v_2'^2 = 0 \)

\[ v_2' = \frac{8 v_1' \pm \sqrt{(8 v_1')^2 - 4(5)(-3 v_2'^2)}}{2(5)} \]

\( \Theta \to 4.88 \text{ m/s} \) not valid by \( v_2' < 0 \)
Problem 2 Solution (Cont'd)

\[ v_2' = 4.88 \text{ m/s} \text{ into Eq. } 3 \]

\[ 4(2.00 \text{ m/s})^2 + (3.70 \text{ m/s})^2 = 4 \cdot v_1'^2 + (4.88 \text{ m/s})^2 \]

\( \therefore v_1' = \boxed{1.21 \text{ m/s}} \)

From Eq. 1

\[ \cos \theta = \frac{v_2'}{4 \cdot v_1'} = \frac{3.70 \text{ m/s}}{4(1.21 \text{ m/s})} = 0.764 \]

\[ \theta = \boxed{40.2^\circ} \text{ (or } 40.1175\ldots \text{ for exact answer)} \]
Problem 2: (25 points) – Momentum conservation

Two billiard balls of masses $m_1 (= 4M)$ and $m_2 (= M)$ move at right angles and meet at the origin of an $xy$ coordinate system. One is moving upward along the $y$ axis at 2.00 m/s, and the other is moving to the right along the $x$ axis with speed 3.70 m/s. After the collision (assumed elastic), the second ball is moving along the positive $y$ axis. (a) What is the final direction of the first ball? What are their two speeds? (b) If $m_1$ is heavier, explain how the final direction is changed.

\[ v_2 = 3.7 \text{ m/s} \]
\[ v_1 = 2.0 \text{ m/s} \]

\[ \begin{align*}
\text{(b) Let } r &= \frac{m_1}{m_2}. \\
\hline
\text{(1') } v_2' &= r v_1' \cos \theta \\
\text{(2') } r v_1' &= r v_1' \sin \theta + v_2' \\
\text{(3') } r v_1'^2 + v_2'^2 &= r v_1'^2 + v_2'^2 \\
\text{(4') } r^2 v_1'^2 &= v_2'^2 + (r v_1' - v_2')^2 \\
\text{(5') } r^2 v_1'^2 &= r^2 v_1'^2 + r v_2'^2 - r v_2'^2 \\
\end{align*} \]

\[ \begin{align*}
\text{\textcircled{4}} \text{'} \text{, } \text{\textcircled{5}} \text{'} &\implies v_2'^2 + (r v_1' - v_2')^2 = r^2 v_1'^2 + r v_2'^2 - r v_2'^2 \\
\therefore (r+1) v_2'^2 - (2 r v_1') v_2' - (r-1) v_2'^2 &= 0 \\
\therefore v_2' &= \frac{2 r v_1' \pm \sqrt{(2 r v_1')^2 + 4 (r+1) (r-1) v_2'^2}}{2(r+1)} \\
\text{\textcircled{4}} \text{'} &\implies v_1' \pm \sqrt{v_1'^2 + v_2'^2} - v_2'^2 \\
\therefore v_1' &= \frac{v_2'}{r} \implies v_1' \to \infty \\
\text{\textcircled{5}} \text{'} &\implies v_1'^2 = v_1'^2 + \frac{v_2'^2}{r} - \frac{v_2'^2}{r} \\
\end{align*} \]

\[ \therefore v_1' = \frac{v_2'}{r} \implies v_1' \to \infty \]

\[ \text{\textcircled{6}} \text{'} \implies \cos \theta = \frac{v_2'}{r v_1'} \implies \frac{v_2'}{r} \to \infty \]

\[ \therefore \theta = 90^\circ \]

If $m_1$ is heavier, $v_1'$ approaches to $v_1' = 6.21 \text{ m/s}$ and $\theta$ does to $90^\circ$ (closer to $+y$ axis).
Problem 3: (25 points) – Newton’s 2nd law

A light cord connected at one end to a block (mass \( m = 3.00 \) kg) which can slide on an inclined plane has its other end wrapped around a pulley. The pulley is a uniform solid cylinder (mass \( M = 30.0 \) kg, radius \( R = 0.200 \) m) resting in a depression at the top of the plane. Both coefficients of static and kinetic friction between the block and the incline are the same and it is \( \mu = 0.0350 \). The magnitude of the acceleration due to the Earth’s gravity is \( g = 9.80 \text{ m/s}^2 \).

(a) Determine the speed of the block after it has traveled \( d = 2.00 \) m along the plane, starting from rest, if the pulley bearing is frictionless.

(b) In fact, it is found that if the block is given an initial speed of \( 0.200 \) m/s, it slides on the plane at the same speed (= \( 0.200 \) m/s). What is the average frictional torque on the pulley?

\[
\begin{align*}
\text{Newton's 2nd Law} & \quad m \frac{d^2\vec{x}}{dt^2} = \sum \vec{F} \quad \left\{ \begin{array}{l}
\vec{F} = ma \\
mg \sin 30^\circ - F_T - F_{fr} = ma_x \quad (1) \\
F_N - mg \cos 30^\circ = 0 \quad (2) \\
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\vec{I} = IA_x & \quad \text{(Moment about the pulley)} \\
F_T R = \left( \frac{1}{2} MR^2 \right) \left( \frac{a_x}{R} \right) \quad \left( \begin{array}{c}
F_T = \frac{1}{2} Ma_x \quad (3)
\end{array} \right)
\end{align*}
\]

Solving Eqs. (1) & (3) to find \( a_x \).

\[
\begin{align*}
mg \sin 30^\circ - \frac{1}{2} Ma_x - \mu mg \cos 30^\circ = ma_x \\
\therefore a_x = \frac{mg(\sin 30^\circ - \mu \cos 30^\circ)}{m + \frac{1}{2} M} = 0.767 \text{ m/s}^2
\end{align*}
\]

\[
\begin{align*}
\text{Using } \quad v_x^2 - v_{x_0}^2 &= 2a_x (x - x_0) \quad \text{and } \quad v_x = \sqrt{2ad} = 1.75 \text{ m/s}
\end{align*}
\]

(b) Modify Eq. (3): \( F_T R - \frac{F_{fr}}{R} = \left( \frac{1}{2} MR^2 \right) \left( \frac{a_x}{R} \right) \quad (4) \)

\[
\begin{align*}
1 \& 4 & \rightarrow mg \sin 30^\circ - \frac{F_{fr}}{R} - \frac{1}{2} Ma_x - \mu mg \cos 30^\circ = ma_x
\end{align*}
\]

\[
\begin{align*}
a_x = 0 \quad \rightarrow \quad \frac{F_{fr}}{R} &= mg \sin 30^\circ - \mu mg \cos 30^\circ \rightarrow F_{fr} = 2.76 \text{ N} \cdot \text{m}
\end{align*}
\]
Problem 4: (25 points) Rigid Body Rotation

You (mass 70.0 kg) stands on a lightweight rotating platform with arms at the your sides, and then the arms are outstretched.

(a) (5 pts) Estimate the moment of inertia \( I_a \) of you (with arms at the sides) using the following approximations: the body (including head and legs) is a solid cylinder (mass \( M = 60.0 \) kg), \( R = 0.120 \) m in radius and \( H = 1.70 \) m high; and each arm is a thin rod (mass \( m = 5.00 \) kg), \( l = 0.600 \) m long, attached to the cylinder.

\[ I_a = \left( \frac{1}{2} MR^2 \right) + \left( 2mR^2 \right) = 0.576 \text{ kg} \cdot \text{m}^2 \]

(b) (5 pts) Using the same approximations, estimate the moment of inertia \( I_b \) when the arms are outstretched.

\[ I_b = \left( \frac{1}{2} MR^2 \right) + \frac{1}{2} \left( \frac{1}{12} ml^2 + m \left( \frac{1}{2} l + R \right)^2 \right) \]

\[ = 2.50 \text{ kg} \cdot \text{m}^2 \]

(c) \( I_a \omega_a = I_b \omega_b \)

\[ \omega_b = \frac{I_a}{I_b} \left( \frac{2\pi}{T_a} \right) = 0.965 \text{ rad/s} \]

\[ T_b = \frac{2\pi}{\omega_b} = 6.51 \text{ s} \]

(d) \( K_b - K_a = \frac{1}{2} I_b \omega_b^2 - \frac{1}{2} I_a \omega_a^2 = -3.89 \text{ J (loss)} \)

(e) \( K_b < K_a \)

It will be easier to lift your arms when rotating because of \( K_b < K_a \).