Chapter 15, example problems:

(15.04) Ultrasound imaging. (Frequency > 20,000 Hz) \( v = 1500 \text{ m/s} \). Frequency required > \( \frac{1500 \text{ m/s}}{1.0 \text{ mm}} = 1.50 \times 10^6 \text{ Hz} \).

(Smaller wave length implies larger frequency, since their product, being equal to the sound velocity, can not change.)

(15.12) Speed of propagation vs. particle speed. Eq. (15.3):

\[ y(x, t) = A \cos \left[ \omega \left( \frac{x}{v} - t \right) \right] = A \cos \left[ 2\pi f \left( \frac{x}{v} - t \right) \right]. \]

(a) Putting in \( \omega = v k \), \( k = \frac{2 \pi}{\lambda} \), and using the fact that \( v \left( \frac{x}{v} - t \right) = x - vt \), we obtain:

\[ y(x, t) = A \cos \left( \frac{2 \pi}{\lambda} \left( x - vt \right) \right). \]

(b) The transverse velocity of a particle in the string on which the wave travels:

The transverse displacement of a particle at position \( x \) on that string as a function of \( t \) is \( y(x, t) = A \cos \left( \frac{2 \pi}{\lambda} \left( x - vt \right) \right) \), so the transverse velocity of that particle as a function of \( t \) is:

\[ v_y(x, t) = \frac{\partial}{\partial t} \{ A \cos \left( \frac{2 \pi}{\lambda} \left( x - vt \right) \right) \} = -A \sin \left( \frac{2 \pi}{\lambda} \left( x - vt \right) \right) \times \left( \frac{2 \pi}{\lambda} \right) \times (-v) = \left( \frac{2 \pi}{\lambda} \right) A \sin \left( \frac{2 \pi}{\lambda} \left( x - vt \right) \right). \]

(c) The maximum transverse speed of a particle in the string, \( v_y^{\text{max}} \), is just the amplitude of the above “velocity wave”, namely \( 2\pi f A \) (\( = \omega A \)). (This is the same expression as that for the maximum velocity of a harmonic oscillator. This is not a coincidence. Every point in a wave is indeed executing a simple harmonic motion, only with a phase which varies with \( x \).) This maximum transverse velocity of a particle in the string is in general different from the wave velocity. For them to be equal, we would have to require \( \omega A = v_x \) or, since \( \omega = v k \), we would have to require \( kA = 1 \), which just means \( A = \frac{\lambda}{2\pi} \). For this maximum transverse velocity to be less or greater than the wave velocity, we need \( A \) to be less or greater than \( \frac{\lambda}{2\pi} \). (In reality, \( A \) is practically always much less than \( \frac{\lambda}{2\pi} \). Thus the maximum transverse velocity of a particle in the string is practically always much less than the wave velocity.)

(15.16) 1.50 m string, weight 1.25 N. Tied to the ceiling. Lower end supports a weight \( W \). Plucked slightly. Wave traveling up the string obeys:

\[ y(x, t) = (8.50 \text{ mm}) \cos(172 \text{ m}^{-1} x - 2730 \text{ s}^{-1} t) \] (thus \( x \) is measured upward).

(a) Time for a pulse to travel the full length of the string:

We need the wave velocity, which is \( 2730 \text{ s}^{-1} / 172 \text{ m}^{-1} = 15.87 \text{ m/s} \). Then the travel time is \( 1.50 \text{ m} / 15.87 \text{ m/s} = 0.0945 \text{ s} \).

(b) Actually, this problem is correct only if the weight of the string, 1.25 N, is much less than the weight \( W \), and therefore can be neglected with respect to the latter. Then the tension in the string, \( F \), is due to \( W \) alone, and is therefore equal to \( W \).

The linear mass density of the string, \( \mu \), is \( (1.25 \text{ N} / 9.8 \text{ m/s}^2) / 1.50 \text{ m} = 0.0850 \text{ kg/m} \). Then using the formula for the velocity of a transverse wave on a string, \( v = \sqrt{\left( F / \mu \right)} \), we can find \( W = v^2 \mu = (15.87 \text{ m/s})^2 \times 0.0850 \text{ kg/m} = 21.41 \text{ N} \), which is indeed much larger than 1.25 N, confirming the validity of the approximation we made in the beginning. (But the error is about 5%).
(c) The number of wavelengths on the string at any instant of time = 
\[ \frac{1.50 \text{ m}}{2\pi / 172 \text{ m}^{-1}} = 41.1 \].

(d) Equation for waves traveling down the string is:
\[ y(x, t) = A \cos \left[ \frac{2\pi}{\lambda} (x + 1587.2 \text{ m/s} \cdot t) \right] \cdot 
\]
(That is, \( A \) and \( \lambda \) can be different for different waves, but the velocity of the waves must all be the same 1587.2 m/s, since it is determined by the tension and the linear mass density of the string.)

(15.22) Threshold of pain. Intensity of sound is 0.11 W/m\(^2\) at 7.5 m from a point source. At what distance from the point source will intensity reach the threshold of pain, 1.0 W/m\(^2\)? Denote the new distance \( D \). Then
\[ 1.0 \text{ W/m}^2 \times 4\pi D^2 = 0.11 \text{ W/m}^2 \times 4\pi (7.5 \text{ m})^2 = \text{power output of the point source}. \]
Hence \( D = 7.5 \text{ m} \times (0.11 \text{ W/m}^2 / 1.0 \text{ W/m}^2)^{1/2} = 2.487 \text{ m} \).
(Note: \( 4\pi D^2 \) is the area of a sphere of radius \( D \).)
Thus you have to move 7.5 m - 2.487 m = 5.013 m toward the source of sound to reach the threshold of sound.

(15.28) Interference of triangular pulses. \( v = 2.00 \text{ cm/s} \).

\[ 2.00 \text{ cm/s} \times 0.25 \text{ s} = 0.5 \text{ cm}. \] The two pulses will barely touch after traveling for 0.25 s, producing:

\[ 2.00 \text{ cm/s} \times 0.50 \text{ s} = 1.0 \text{ cm}. \] The two pulses now overlapped by 1.00 cm .

The superposed pulses now show a flat top for a width of 1.00 cm .
2.00 cm/s × 0.75 s = 1.5 cm. The two pulses now totally overlap. Their superposition gives one triangular pulse of twice the height, but the same old width of 2.00 cm. But this shape is maintained for only one instant of time, since part of it is moving to the right, and part of it is moving to the left.

2.00 cm/s × 1.00 s = 2.00 cm. The two pulses have now moved pass each other by 1.0 cm. The superposition again produces the trapezoidal shape with a flat top of 1.00 cm just like when it has only moved for 0.5 s. But notice the directions of the two horizontal arrows, which indicate the directions of motion of the two constituent pulses.

2.00 cm/s × 1.25 s = 2.50 cm. The two pulses have now moved pass each other by their own width, so they no longer overlap. The situation appears just like when they have moved for only 0.25 s, except for the directions of the two horizontal arrows, which indicate the directions of motion of the two constituent pulses.
(15.34) Distance between adjacent antinodes of a standing wave is 15.0 cm. A particle at an antinode oscillates in SHM with amplitude 0.850 cm and period 0.0750 s. The string is along x and is fixed at x = 0.

(a) Adjacent nodes are also 15.0 cm apart.

(b) Recall that \( \cos (A - B) + \cos (A + B) = 2 \cos A \cos B \), which implies that \( A \cos (kx - \omega t) + A \cos (kx + \omega t) = 2 A \cos (kx) \cos (\omega t) \). Hence the amplitude of a standing wave is a factor two larger than the amplitude of either of the two constituent traveling waves, but their wavelengths are the same, and their frequencies are the same. Thus: Wavelength of each traveling wave \( \lambda = 30.0 \text{ cm} \); Amplitude of each traveling wave \( A = 0.425 \text{ cm} \). Speed of each traveling wave \( v = \lambda f = \lambda / T = 30.0 \text{ cm} / 0.075 \text{ s} = 400 \text{ cm/s} = 4 \text{ m/s} \).

(c) Maximum transverse speed of a point at the antinode of the standing wave is \( A \omega = A (2\pi / T) = 0.850 \text{ cm} (2\pi / 0.0750 \text{ s}) = 71.21 \text{ cm/s} \). Minimum speed = 0.

(d) Shortest distance between a node and antinode is 7.50 cm.

(15.54) Weightless ant. An ant of mass \( m \) stands on top of a horizontal stretched rope. Rope has mass per unit length \( \mu \) and tension \( F \). A sinusoidal transverse wave of wave length \( \lambda \) propagates on the rope. Motion is in the vertical plane. Find the minimum wave amplitude which will make the ant momentarily weightless. The mass \( m \) is so small that it will not affect the propagating wave.

The ant will become momentarily weightless when it momentarily needs no support from the rope. This happens when its mass times the maximum downward acceleration of the SHM of the ant is exactly equal to its weight. This happens when the ant is at the highest point of its SHM. [Then just before or after this moment, the acceleration will still be downward but with a magnitude less than this maximum value, and the weight of the ant will need to be partially cancelled by an upward support force from the rope, so that the net downward force is still equal to (now smaller) \( ma \). Thus the weightlessness feeling happens only at that brief moment when the ant is at the highest point of the SHM. During the half cycle of the motion when the acceleration is pointing upward, the weight is pointing in the wrong direction to provide the needed force. So an upward support force from the rope larger than the weight of the ant will exist, so that the net force is upward, and is still equal to \( ma \). Thus in this half cycle the ant actually weights more than its actual weight, if you put a scale between the ant and the rope to measure this apparent weight.]

The magnitude of the maximum acceleration of the SHM is \( m\omega^2 \). But \( \omega = v k = \sqrt{(F / \mu) (2\pi / \lambda)} \). Thus we must require \( mA(g / \mu) (2\pi / \lambda)^2 = mg \), giving \( A = (\mu g / F) (\lambda / 2\pi)^2 \). Let us check the unit. The unit of \( \mu g \) is N/m. The unit of \( (\mu g / F) \) is therefore just 1/m. The unit of \( (\lambda / 2\pi)^2 \) is just m\(^2\). So their product has the unit of m, which is the right unit for \( A \).
(15.62) More general sinusoidal wave: \( y(x, t) = A \cos (kx - \omega t + \phi) \).

(a) \( \text{At } t = 0, \ y(x, t = 0) = A \cos (kx + \phi) \).

For \( \phi = 0 \),
\[ y(x, 0) = A \cos (kx) : \]

For \( \phi = \pi/4 \),
\[ y(x, 0) = A \cos (kx + \pi/4) : \]
That is, even at \( x = 0 \),
\( y \) is already \( \cos (\pi/4) \).

For \( \phi = \pi/2 \),
\[ y(x, 0) = A \cos (kx + \pi/2) : \]
That is, even at \( x = 0 \),
\( y \) is already \( \cos (\pi/2) \).
From this figure, we can see that it is the same as:
\[ y(x, 0) = -A \sin (kx) . \]
The wave is then given by:
\[ y(x, 0) = -A \sin (kx - \omega t) . \]

For \( \phi = 3\pi/4 = \pi/2 + \pi/4 \),
\[ y(x, 0) = A \cos (kx + 3\pi/4) : \]
That is, even at \( x = 0 \),
\( y \) is already \( \cos (3\pi/4) \).

For \( \phi = 3\pi/2 = \pi + \pi/2 \),
\[ y(x, 0) = A \cos (kx + 3\pi/2) : \]
That is, even at \( x = 0 \),
\( y \) is already \( \cos (3\pi/2) \).
From this figure, we can see that it is the same as:
\[ y(x, 0) = +A \sin (kx) . \]
The wave is then given by:
\[ y(x, 0) = +A \sin (kx - \omega t) . \]
(b) Transverse velocity: \( v_y = \frac{\partial y}{\partial t} = + A \omega \sin (kx - \omega t + \phi) \).

(c) At \( t = 0 \), a particle at \( x = 0 \) has displacement \( y = A / \sqrt{2} \). Can one determine \( \phi \)? No. There are several candidates: At \( t = x = 0 \), \( y(0, 0) = + A \cos (\phi) \).

We first calculate the transverse velocity \( v_y = \frac{\partial y}{\partial t} = + A \omega \sin (- \omega t + \phi) \), and evaluate it at \( t = 0 \), and get \( v_y(x = 0, t = 0) = + A \omega \sin (\phi) \). At \( \phi = \pi/4 \), this transverse velocity is positive, so it is not moving toward \( y = 0 \). So we reject it. At \( \phi = 7\pi/4 \), this transverse velocity is negative, so it is moving toward \( y = 0 \). So we accept this answer. Hence we conclude that \( \phi = 7\pi/4 \) (or, equivalently, \(-\pi/4\)).

Can one get this answer without doing so much math? Yes! First, one should realize that if \( y(x, t) = A \cos (kx - \omega t + \phi) \), then the wave is moving toward positive \( x \), or to the right in our plots. Then looking at our plot for \( \phi = 7\pi/4 \), and let the curve move to the right. One will find that \( y \) will increase at \( x = 0 \). So we should reject this answer. Then we do the same thing for \( \phi = 7\pi/4 \) or \(-\pi/4\), and see that it can be accepted.

(d) In general, to determine \( \phi \), we need to know:

(i) \( y(x = 0, t = 0) \); and (ii) the sign of \( v_y(x = 0, t = 0) \).

(15.68) Vibrating string. 50.0 cm long. Tension \( F = 1.00 \) N. Five stroboscopic pictures shown. Strobo rate is 5000 flashes per minute. Maximum displacement occurs at flashes 1 and 5, with no other maxima in between.

(a) Period: \( T = 0.0016 \) min = 0.096 s (the time for 8 flashes).

Frequency \( f = 1 / T = 10.42 \) Hz.

Wavelength = 50.0 cm. All for either of the two the traveling wave on this string that are moving in opposite directions to form this standing wave.

(b) The second harmonic (also known as the first overtone).

(c) Speed of the traveling waves on this string

\( v = \frac{\lambda}{T} = 50.0 \) cm / (0.096 s) = 520.8 cm/s = 5.21 m/s.

(d) Point P. (i) In position 1 it is moving with a transverse velocity of \( v_y = 0 \).

(ii) In position 3 it is moving with a transverse velocity of \( v_y = A \omega = A(2\pi f) = \)
3.0 cm × (2π × 10.42 s⁻¹) = 3.0 cm × 65.47 s⁻¹ = 196.4 cm/s = 1.96 m/s.

(c) Mass of this string \( m = 0.50 \text{ m} \times \mu = 0.50 \text{ m} \times [1.00 \text{ N} / (5.21 \text{ m/s})^2] = 0.01842 \text{ kg} = 18.42 \text{ g}. 

(15.74) String, Both ends held fixed. Vibrates in the third harmonic. Speed 192 m/s. Frequency 240 Hz. Amplitude at an antinode is 0.400 cm.

\[ \lambda = \frac{192 \text{ m/s}}{240 \text{ Hz}} = 0.8 \text{ m} \]

(a) Find amplitude of oscillation:

(i) at \( x = 40.0 \text{ cm} \) from the left end of the string.

The general equation for the standing wave is:

\[ y(x, t) = 0.400 \text{ cm} \times \sin (2\pi x / 0.8 \text{ m}) \times \sin (2\pi \times 240 \text{ Hz} \times t) \]

We let \( x = 0 \) be the left end of the string. We should have the factor \( \sin (2\pi x / 0.8 \text{ m}) \) because the left end is a node, and \( \sin 0 = 0 \). Also, the wavelength is 0.8 m, and so at \( x = 0.8 \text{ m} \) we should get \( \sin (2\pi) = 0 \).

So the amplitude of transverse oscillation at \( x = 40.0 \text{ cm} \) is:

\[ 0.400 \text{ cm} \times \sin (2\pi \times 0.40 \text{ cm} / 0.8 \text{ m}) = 0 \]

(ii) at \( x = 20.0 \text{ cm} \) from the left end of the string, the amplitude is:

\[ 0.400 \text{ cm} \times \sin (2\pi \times 0.20 \text{ cm} / 0.8 \text{ m}) = 0.400 \text{ cm} \]

(iii) at \( x = 10.0 \text{ cm} \) from the left end of the string, the amplitude is:

\[ 0.400 \text{ cm} \times \sin (2\pi \times 0.10 \text{ cm} / 0.8 \text{ m}) = 0.283 \text{ cm/s} \]

(b) At each point of part (a), find time to go from largest upward displacement to largest downward displacement. The answer is the same, and is equal to: \( T / 2 \), or \( \frac{1}{2f} = 0.00208 \text{ s} \), except at the nodes of the standing wave, i.e., at \( x = 40.0 \text{ cm} \) in the three cases of part (a).

(c) Maximum transverse velocity of the motion:

\[ v_y(x, t) = \frac{\partial y(x, t)}{\partial t} = 0.400 \text{ cm} \times (2\pi \times 240 \text{ Hz}) \times \sin (2\pi x / 0.8 \text{ m}) \times \cos (2\pi \times 240 \text{ Hz} \times t) \]

The amplitude of transverse velocity at distance \( x \) from the left end is

\[ 0.400 \text{ m} \times (2\pi \times 240 \text{ Hz}) \times \sin (2\pi x / 0.8 \text{ m}) \]

So at \( x = 40.0 \text{ cm} \) from the left end of the string, the maximum transverse velocity is:

\[ 0.400 \text{ cm} \times (2\pi \times 240 \text{ Hz}) \times \sin (2\pi \times 0.40 \text{ m} / 0.8 \text{ m}) = 0 \]

at \( x = 20.0 \text{ cm} \) from the left end of the string, the maximum transverse velocity is:

\[ 0.400 \text{ cm} \times (2\pi \times 240 \text{ Hz}) \times \sin (2\pi \times 0.20 \text{ m} / 0.8 \text{ m}) = 603.2 \text{ cm/s} = 6.03 \text{ m/s} \]

at \( x = 10.0 \text{ cm} \) from the left end of the string, the maximum transverse velocity is:

\[ 0.400 \text{ cm} \times (2\pi \times 240 \text{ Hz}) \times \sin (2\pi \times 0.10 \text{ m} / 0.8 \text{ m}) = 426.5 \text{ cm/s} = 4.27 \text{ m/s} \]

Maximum transverse acceleration of the motion:

\[ a_y(x, t) = \frac{\partial v_y(x, t)}{\partial t} = -0.400 \text{ cm} \times (2\pi \times 240 \text{ Hz})^2 \times \sin (2\pi x / 0.8 \text{ m}) \times \sin (2\pi \times 240 \text{ Hz} \times t) \]

The amplitude of transverse acceleration at distance \( x \) from the left end is

\[ 0.400 \text{ m} \times (2\pi \times 240 \text{ Hz})^2 \times \sin (2\pi x / 0.8 \text{ m}) \]

So at \( x = 40.0 \text{ cm} \) from the left end of the string, the maximum transverse acceleration is:

\[ 0.400 \text{ m} \times (2\pi \times 240 \text{ Hz})^2 \times \sin (2\pi \times 0.40 \text{ m} / 0.8 \text{ m}) = 0 \]

at \( x = 20.0 \text{ cm} \) from the left end of the string, the maximum transverse acceleration is:

\[ 0.400 \text{ m} \times (2\pi \times 240 \text{ Hz})^2 \times \sin (2\pi \times 0.20 \text{ m} / 0.8 \text{ m}) = 0 \]

at \( x = 10.0 \text{ cm} \) from the left end of the string, the maximum transverse acceleration is:

\[ 0.400 \text{ m} \times (2\pi \times 240 \text{ Hz})^2 \times \sin (2\pi \times 0.10 \text{ m} / 0.8 \text{ m}) = 0 \]
603.2 cm/s = 909604 cm/s^2 = 9096 m/s^2.

at x = 10.0 cm from the left end of the string, the maximum transverse acceleration is:

\[0.400 \text{ cm} \times (2\pi \times 240 \text{ Hz})^2 \times \sin (2\pi \times 0.10 \text{ m} / 0.8 \text{ m}) = 643147 \text{ cm/s}^2 = 6431 \text{ m/s}^2.\]

Actually, these answers are easy to see by looking at the figure:

The second node from the left end is at 0.8 m, because you see a whole wavelength between here and the left end. Then the first node from the left end must be at 0.4 m. The first antinode from the left end must be at 0.2 m. This is why at x = 0.2 m we find the largest amplitudes of velocity and acceleration, and at x = 0.4 m we find vanishing amplitudes of velocity and acceleration. In addition, the amplitudes of velocity and acceleration at x = 0.1 m is down from those at x = 0.2 m by a factor \(\sin (\pi/4) = 1/\sqrt{2} = 0.707\), as the \(x\)-dependent factor is simply a sine function to give \(\sin (\pi/2) = 1\) at x = 0.2 m, and to give \(\sin (\pi) = 0\) at x = 0.4 m. It will also give 0 at x = 0.8 m, and at x = 1.2 m, which is the right end of the string. They correspond to \(\sin (2\pi)\) and \(\sin (3\pi)\) from the \(x\)-dependent factor.