1. The essential properties of a periodic motion
(also known as oscillation, vibration, or oscillatory motion, vibrational motion)

A back-and-forth motion which repeats itself. (If the oscillation gradually
dies down, that is, if its range of excursion diminishes in time, it is called a
damped oscillation or damped vibration.)

cycle – one complete back-and-forth motion, returning to the starting point.
frequency \((f)\) – the number of cycles per second. (Unit: Hz, i.e., hertz.)
\(1 \text{ Hz} = 1 \text{ s}^{-1}\).
period \((T)\) – The time it takes to complete one cycle.

We have: \(f = 1/T\) or \(T = 1/f\).

Simple harmonic motion (SHM) – A particularly regular and simple type
of oscillation: Its displacement is sinusoidal, meaning that its displacement
obeys:

\[ x(t) = A \cos(\omega t + \phi), \]

where \(\omega = 2\pi f\) is the angular frequency (unit: radians per second), and \(A\)
is the amplitude of the SHM, and \(\phi\) its phase (or phase angle) at \(t = 0\).

Note that for \(\phi = -\pi/2\), it becomes \(x(t) = A \sin(\omega t)\), and for \(\phi = 0\), it be-
comes: \(x(t) = A \cos(\omega t)\). Also, for \(\phi = \pi/2\), it becomes \(x(t) = -A \sin(\omega t)\),
and for \(\phi = \pi\), it becomes: \(x(t) = -A \cos(\omega t)\). All these conclusions require
no memory: They can be read off from a figure of the cosine function
(which begins with unity):

For a SHM, we also have, for the velocity \(\nu\) of the SHM at any time \(t\):

\[ \nu(t) = dx(t)/dt = -A \omega \sin(\omega t + \phi), \]
and for the acceleration $a$ of the SHM at any time $t$:

$$a(t) = \frac{d^2x(t)}{dt^2} = -A\omega^2 \cos(\omega t + \phi),$$

Note that $a(t)$ obeys: $a(t) = -\omega^2 x(t)$. See below why this should be true.

To arrive at the above two equations, all one needs to use is the chain rule in calculus: $df(g(x))/dx = [df(g)/dg]_{g=g(x)}[dg(x)/dx]$, and the two simple equations $d\sin(x)/dx = \cos(x)$, and $d\cos(x)/dx = -\sin(x)$, which require no memory, as they can be read off from the behaviors of $\sin(x)$ and $\cos(x)$, if you know that derivative of a function is just the slope of the curve obtained by plotting the function.

In order for a mass $m$ to execute a SHM, it has to receive a harmonic restoring force, which means a force with a direction opposite to the direction of $x$, and a magnitude proportional to the magnitude of $x$. That is, the force must obey the equation:

$$F = -kx,$$

where $k$ is called the force constant, or spring constant, since a spring, with one end fixed, can provide such a force to a mass attached to its other end, if $x$ is measured from the equilibrium position of the mass (where the mass can sit at rest). Usually, extending the length of the spring from its equilibrium length is defined as positive $x$. Then compressing the length of the spring from its equilibrium length corresponds to negative $x$.

With such force acting on a mass $m$, Newton's law would give the differential equation:

$$m \frac{d^2x}{dt^2} = \sum F_x = -kx.$$

It has special solutions $\sin \omega t$ and $\cos \omega t$, where $\omega = \sqrt{k/m}$, which can be combined to obtain the general solution:

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t = A \cos(\omega t + \phi),$$

where we have set $C_1 = A \cos \phi$, and $C_2 = -A \sin \phi$, and used the math identity $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$. This form for $x(t)$ defines a SHM.
The amplitude $A$, and the initial phase $\phi$ can be determined by the initial values of $x$ and $v$. That is, the values of $x$ and $v$ at time $t = 0$, which are also written as $x(0)$ and $v(0)$, or $x_0$ and $v_0$.

Note that any SHM can be viewed as an effective spring, with an effective spring constant $k$, driving an effective mass $m$ to execute a SHM. However, unless $m$ or $k$ is given, otherwise only the ratio $k/m$ can be determined from the SHM (viz., from its $\omega$ or $f$), and not the individual $m$ and $k$.

In an angular SHM, mass $m$ is replaced by moment of inertia $I$, restoring force $F$ is replaced restoring torque $\tau$, displacement $x$ is replaced by angular displacement $\theta$, force constant is replaced by angular force constant $K$ (so that $\tau = -K\theta$), velocity $v$ is replaced by angular velocity $d\theta/dt$, and acceleration $a$ is replaced by angular acceleration $\alpha$. but $\omega (= 2\pi f)$ remains the same, and it should not be confused with $d\theta/dt$, which in rotational motion is denoted as $\omega$, but not here! (This is because we shall soon introduce the angular velocity $\omega$ of a circular motion, which is directly related to the $\omega$ in the SHM.)

A linear SHM can be viewed as the motion of the shadow of a vertical circular motion, when light shines on the moving object from straight above. The $\omega$ of the SHM is just the angular velocity of this circular motion. [A circular motion is described by two equations: $x = A \cos(\omega t + \phi)$ and $y = \pm A \sin(\omega t + \phi)$, with the $\pm$ sign decided by its direction of motion (counterclockwise or clockwise). The projection of either circular motion onto the $x$ axis gives the first equation alone, which is a SHM.]

In angular SHM there is $d\theta/dt$ which ordinarily would be denoted as $\omega$, but it is not denoted as $\omega$ here, since $\omega$ is already defined here to mean something else, namely, $2\pi f$. An angular SHM is not the shadow of any motion.

For a linear SHM the displacement $x$, velocity $v$, and acceleration $a$ are all constantly changing with time. Both $|x|$ and $|a|$ reach their peak values at the end points of the excursion, whereas $|v|$ reaches its peak value whenever the motion passes through its center, whether from the right side or from the left side. The sign of $a$ is always opposite to that of $x$. That is, whenever $x$ is positive, $a$ is negative, and whenever $x$ is negative, $a$ is positive. On the other hand, the motion is always speeding up when it is moving toward the center, whether from the right side or from the left side, and it is always slowing down when it is moving away from the center.
2. **Energy of a SHM**

The total energy \( E \) of a SHM is always the sum of two parts: a kinetic energy part \( (K) \) and a potential energy part \( (U) \). Their sum is a constant of time, but each part changes with time because \( x \) and \( v \) change with time. These changes are periodic, and repeat themselves. — When \( U \) is going up, \( K \) is going down, and when \( U \) is going down, \( K \) is going up, — without changing their sum. At the end points of the SHM, \( K \) is zero, (because the velocity \( v \) is zero there), and the energy is pure \( U \). At the center of the excursion, \( U \) is zero there, and the energy is pure \( K \).

\[
K = \frac{1}{2} mv^2, \quad U = \frac{1}{2} kx^2, \quad \text{and} \quad E = K + U.
\]

Thus, \( E = (1/2)kA^2 \) at the end points of the excursion, and \( E = (1/2)m(v_{\text{max}})^2 \) at the center of the excursion. Since \( v_{\text{max}} \equiv \omega A \) for the magnitude of the maximum velocity, occurring when the motion passes through its center, the two \( E \) expressions are equal, as is required by the fact that \( E \) is a constant in time, also known as a constant of motion.

Since \( E \) is a constant in time, we can also conclude

\[
\frac{1}{2} m[u(t)]^2 + \frac{1}{2} k[x(t)]^2 = \frac{1}{2} kA^2,
\]

so

\[
u(t) = \pm \omega \sqrt{A^2 - [x(t)]^2} = \pm v_{\text{max}} \sqrt{1 - [x(t)/A]^2},
\]

which is true at any time moment of the motion!

3. **Simple pendulum and physical pendulum**

If a mass \( m \) is hung at the lower end of a string of length \( L \) with the upper end of the string fixed, then the mass can swing about its lowest position. This is called a pendulum. This is an angular motion. So it satisfies the angular Newton's law: \( I\alpha = \Sigma \tau \), where \( I = mL^2 \), and \( \alpha = d^2 \theta / dt^2 \). Here there is only one torque, which is due to the gravitational force \( mg \) pulling the mass downward. The lever arm is \( L \sin \theta \), so \( \tau = -mgL \sin \theta \), where the minus sign is added because the torque is in the direction to reduce \( \theta \), when
\( \theta \) is positive. (A torque is positive if it is counter-clockwise, but when \( \theta \) is positive, the torque is seen to be clockwise, and therefore negative. Similarly, when \( \theta \) is negative, the torque is seen to be counter-clockwise, and therefore positive. Thus one can see that the torque and \( \theta \) always differ by a sign.) Putting this torque formula into the angular Newton’s law, we obtain the differential equation:

\[
 mL^2 \frac{d^2 \theta}{dt^2} = -mgL \sin \theta.
\]

This is called the **pendulum equation**. If the maximum \( \theta \) is small, then we can approximate \( \sin \theta \) by \( \theta \) in radians. The equation then reduces to the equation for a SHM:

\[
 m \frac{d^2 x}{dt^2} = -k_{\text{eff}} x, \quad \text{where} \quad k_{\text{eff}} = \frac{mg}{L}, \quad \text{and} \quad x \equiv L \theta.
\]

Note that \( L \theta \) is the curved arc length that the mass has traveled from the lowest position of the mass, which is almost a straight horizontal length when \( \theta \) is small. A pendulum with \( \theta \) never reaching bigger than about 10º is called a **simple pendulum**. It behaves as a simple harmonic oscillator with an effective spring constant \( k_{\text{eff}} \) given above. That is, it behaves as if there is a spring with such a spring constant attached to the mass, even though it actually does not have any spring attached to the mass.

The frequency of a simple pendulum is

\[
 f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k_{\text{eff}}}{m}} = \frac{1}{2\pi} \sqrt{\frac{g}{L}}
\]

is independent of the mass \( m \)! Its period \( T \) is equal to \( 1/f \), which is:

\[
 T = 2\pi \sqrt{\frac{L}{g}}.
\]

Note that the frequency or period of a simple pendulum is independent of its mass \( m \)! That is, two pendulums of the same \( L \) but different \( m \) will oscillate in synchrony, but two pendulums of the same \( m \) but different \( L \) will not!
If we pivot any finite object at any point in the object, it is called a **physical pendulum**. It can also execute a SHM if the swinging angle is small.

Shown on the right is a swinging chair pivoted at the top. The middle position shown is the **equilibrium position** of the chair. It is also the lowest position of its center of mass. But notice that its back is not vertical in this position, because at this position the chair should not receive any torque with respect to the pivot point, so its center of mass must be straight below its pivot point, by a distance \( d \), say. When the chair has swung an angle \( \theta \), as shown, The torque acting on the chair by gravity is \(-mgd \sin \theta\), as if the whole mass of the chair is located at where the center of mass is located. The angular Newton's law still reads \( I \frac{d^2 \theta}{dt^2} = \Sigma \tau = -mgd \sin \theta \), but \( I \) is no longer equal to \( md^2 \) here, which is only valid for a point mass at a distance \( d \) from the pivot point, as is the case of a simple pendulum. If the amplitude of the swing is small, \( \sin \theta \) can be approximated by \( \theta \), the equation again reduces to that of a SHM, except that the effective spring constant now reads

\[
k_{\text{eff}} = \frac{m \cdot (\text{coefficient of } \theta \text{ in } \tau)}{I} = \frac{m^2 gd}{I},
\]

so that for a physical pendulum,

\[
f = \frac{1}{2\pi} \sqrt{\frac{mgd}{I}}, \quad \text{and} \quad T = 2\pi \sqrt{\frac{I}{mgd}}.
\]

A **tortional pendulum** also executes an angular SHM, except that its restoring torque comes from twisting a wire, or a long bar, or a spring. The angular Newton's law still reads \( I \frac{d^2 \theta}{dt^2} = \Sigma \tau \), and the property of a twisted wire or bar still gives \( \tau = -K\theta \), so we again have an angular SHM, with \( \theta \) replacing \( x \), \( I \) replacing \( m \), and \( K \) replacing \( k \). Therefore, for a tortional pendulum,

\[
f = \frac{1}{2\pi} \sqrt{\frac{K}{I}}, \quad \text{and} \quad T = 2\pi \sqrt{\frac{I}{K}}.
\]
4. **Damped harmonic motion**

If air or internal friction is added into the consideration, the governing differential equation for a SHM becomes

\[
m \frac{d^2 x}{dt^2} = -k x - b \frac{dx}{dt},
\]

where the first term on the right side is a *harmonic restoring force*, and the second term on the right side is a type of *friction force*, which is proportional to the velocity of the motion, and its direction is opposite to that of the velocity. (It describes air friction and some types of internal friction.) The differential equation can be rearranged to appear as:

\[
m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + k x = 0,
\]

This *linear differential equation* has a general solution of the form

\[
x(t) = Ae^{-(b/2m)t} \cos \left( \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} t + \phi \right),
\]

with \( A \) and \( \phi \) determined by initial displacement and velocity. This solution is valid only when the quantity inside the square root is positive. That is, only when

\[\frac{k}{m} - \frac{b^2}{4m^2} \geq 0.\]

If the left hand side is indeed strictly > 0, that is, if \( b^2 < 4mk \), the oscillator is said to be **underdamped**. If the left hand side is = 0, that is, if \( b^2 = 4mk \), the oscillator is said to be **critically damped**, at which point the oscillation disappears, and \( x(t) \) simply decays exponentially, as given by the factor \( e^{-(b/2m)t} \). (The cosine factor becomes simply a time-independent constant factor, \( \cos \phi \).) If the left hand side is strictly < 0, that is, if \( b^2 > 4mk \), the oscillator is said to be **overdamped**. The above solution is not valid in this regime, and must be replaced by another form. It also has no oscillation, but it decays in a form that is faster than a simple exponential function.
The frequency of oscillation of an underdamped oscillator is:

\[
f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.
\]

which is lower than the frequency when there is no damping, that is, when \( b = 0 \). Note that \( f' \) drops to zero at critical damping.

5. **Forced vibration (or driven oscillation) and resonance.**

Let a damped oscillator be subject to an external oscillatory force of frequency \( f = \omega/2\pi \), which may or may not be equal to the intrinsic frequency of the oscillator, which we now denote as

\[
f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}.
\]

The governing differential equation now reads:

\[
m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F_0 \cos \omega t.
\]

We are not interested in the general solution of this equation, which is quite complicated. Rather, we are only interested in the so-called **steady-state solution**, which means an oscillatory solution with a constant amplitude, which occurs long time after getting the motion started with an initial displacement and velocity. (The driving force is assumed to exist for all \( t > 0 \). Then this steady solution can be easily obtained by substituting \( x(t) = A\sin(\omega t + \phi_0) \) into the above differential equation and solve for \( A \) and \( \phi_0 \).)

This steady-state solution has the form:

\[
x(t) = \frac{F_0}{m\sqrt{(\omega^2 - \omega_0^2)^2 + b^2 \omega^2 / m^2}} \sin(\omega t + \phi_0),
\]

which is independent of the initial displacement and velocity, but is only dependent on the amplitude \( F_0 \) of the driving oscillatory force, the driving angular frequency \( \omega \) (or frequency \( f = \omega / 2\pi \)), and the intrinsic angular frequency of the oscillator \( \omega_0 \) (or the intrinsic frequency of the oscillator \( f_0 = \omega_0 / 2\pi \)).
In the above expression,

\[ \phi_0 = \tan^{-1} \frac{\omega_0^2 - \omega^2}{\omega (b / m)} , \]

which is not particularly important. The important part is the amplitude of this driven oscillation, which is:

\[ A_0 = \frac{F_0}{\sqrt{m^2 (\omega^2 - \omega_0^2)^2 + b^2 \omega^2 / m^2}} , \]

which is bell-shaped if plotted against \( \omega \), as shown, and its peak, for small \( b \) is essentially located at \( \omega = \omega_0 \). Thus the amplitude of this driven oscillation is largest at this peak frequency, which is called the resonance frequency. (If \( b \) is large the resonance frequency is below \( \omega_0 \).) This phenomenon is called resonance or resonance phenomenon, and it has important consequences. For example, to get a swing to swing at large angles, it is necessary to shake it at almost its resonance frequency, and soldiers marching on a suspension bridge near its resonant frequency might start such a strong swinging motion of the bridge that it might cause the bridge to collapse! Sometimes when an airplane fly by the down town of a city, the large number of window glasses of a tall high-rise building can all shatter at the same time, because all these glasses have the same shape and size, and therefore they all have the same resonance frequency. If the flying-by airplane can generate sound containing this frequency, it can cause all those glasses to start a resonant vibration, with amplitude growing in time, eventually causing all of those glasses to shatter! (The amplitude of a resonant forced harmonic oscillator will grow toward the steady-state value, which is so large for those glasses that shattering occurs before this steady-state amplitude is reached.)

An ac electric loop circuit involving a capacitor, an inductor, and a resistor connected in series has also an intrinsic resonance frequency for electric oscillation. A weak radio wave in the air, broadcasted by a radio (or TV) station, can provide a weak driving term to that circuit. If its frequency is very near the resonance frequency of the circuit, a large electric oscillation can be induced in the circuit. If further amplified, it can drive a loudspeaker so you can hear the broadcast (after rectification) (or even watching some pictures on a TV). The frequency of the broadcasted wave from that particular station is fixed, so one needs to change the capacitance of the capacitor in the resonance circuit in order to achieve resonance. This is what you do
when you tune a radio (or TV). This is an important application of the resonance phenomenon, since it allows you to select a channel and suppress unwanted signals from all other channels (broadcasters) and also any possible noise generated by any nearby electric device which can generate radio waves, (if their frequencies are all outside the width of the bell-shaped resonance curve).

For a radio to be able to distinguish between two very closely spaced frequencies, the bell-shaped resonance curve needs to be tall and narrow. This is when $b$ is small. The width of the bell-shaped peak, $\Delta \omega$, is equal to $b/m$. It needs to be compared with $\omega_0$, in order to see how sharp is the resonance. Hence one defines:

$$Q = \frac{\omega_0}{\Delta \omega} = \frac{m \omega_0}{b}.$$  

It is called the quality factor, or $Q$ value, of a resonance. Clearly, the higher is the $Q$ value, the sharper is the resonance, since it means smaller $\Delta \omega$ with respect to $\omega_0$. Thus a good tuner should have a high $Q$ value. It is an indication of good quality.