THREE DIMENSIONAL RADIAL PROBLEMS IN SPHERICAL POLAR COORDINATES

In terms of the function \( \chi(r) \equiv r R(r) \), the wave equation is

\[
\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + V(r) + \frac{\hbar^2 \ell (\ell + 1)}{2\mu r^2} \right] \chi = E \chi.
\]

This has exactly the form of a 1-D problem with the effective potential energy given by

\[
U(r, \ell) = V(r) + \frac{\hbar^2 \ell (\ell + 1)}{2\mu r^2}.
\]

Also, \( \chi(r) = 0 \) at \( r = 0 \) is equivalent to \( U(r, \ell) = 0 \) for \( r \leq 0 \). Hence the qualitative and quantitative procedures used for the 1-D case can be immediately applied, but it is necessary to treat each value of \( \ell \) as a separate problem. Before proceeding to consider "canonical examples", note the physical significance of the \( \chi(r) \) function.

\[
\psi = R(r) Y(\theta, \phi)
\]

\[
|\psi|^2 = |R|^2 |Y|^2.
\]

If we want the probability that the particle is between \( r \) and \( r + dr \), regardless of angular orientation, we integrate \( |Y|^2 \) factor over all angles and get

\[
\int \int |\psi|^2 r^2 \sin \theta d\theta d\phi dr = 4\pi r^2 |R|^2 dr
\]

\[
4\pi |\chi|^2 dr
\]

so \( |\chi|^2 \) is radial probability distribution.

Hydrogen Atom

Consider

\[
\left( -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} - \frac{e^2}{r} + \frac{\hbar^2 \ell (\ell + 1)}{2\mu r^2} \right) \chi = E \chi.
\]

It is convenient to introduce atomic units:

- distance in terms of Bohr radius, \( a_o = \frac{\hbar^2}{\mu e^2} \)

- energy in terms of Rydberg energy, \( Ry = \frac{e^2}{2a_o} \)

These are equivalent to setting \( \hbar = 1 \), \( e^2 = 2 \), \( \mu = \frac{1}{2} \).
RADIAL PROBLEMS - 2

Thus the problem becomes

\[ \frac{d^2 \chi}{dr^2} + \left( E + \frac{2}{r} - \frac{\ell(\ell+1)}{r^2} \right) \chi = 0 \]

with \( \chi = 0 \) at \( r = 0 \) and \( \chi \to 0 \) as \( r \to \infty \). The bound state energy levels are given by

\[ E = -\frac{1}{2} \text{Unusual case since no dependence on } \ell \text{ in Rydberg units} \]

\[ n = 1, 2, 3, \ldots \]

The qualitative form of the radial wavefunctions can be readily inferred by examining the effective potential function

\[ U(r, \ell) = -\frac{2}{r} + \frac{\ell(\ell+1)}{r^2} \quad r > 0 \]
\[ = \infty \quad r < 0 \]

The classical turning points are defined by

\[ E - U(r, \ell) = \frac{1}{2} \mu r^2 = 0 \text{ at } r = r_0, \text{ where } \dot{r} = 0. \]

Thus,

\[ -\frac{1}{n^2} + \frac{2}{r_0} - \frac{\ell(\ell+1)}{r_0^2} = 0 \quad \text{or } U(r_0, \ell) = -\frac{1}{n^2} \]

or

\[ r_0 = n \left[ n \pm \sqrt{n^2 - \ell(\ell+1)} \right]. \]

The minimum in \( U(r, \ell) \) is given by

\[ \left( \frac{dU}{dr} \right)_r = 0 \quad \text{or } \frac{2}{r_m^2} = 2 \frac{2\ell(\ell+1)}{r_m^3} = 0 \]

and hence \( r_m = \ell(\ell+1) \) and \( U(r_m, \ell) = -\frac{1}{\ell(\ell+1)} \).

With these values for \( r_0, r_m, U(r_0, \ell), U(r_m, \ell) \), we can sketch the \( U(r, \ell) \) functions and the wavefunctions. Roughly, we find:
Note that the nodal properties are apparent from these sketches: the \((n, \ell)\) radial function has \(n - 1 - \ell\) nodes (not counting that at \(r = 0\)). The qualitative form of the wavefunctions reflects the classical turning points and the extremely flat character of \(U(r, \ell)\) at large \(r\):

\[
\ell = \sqrt{\frac{\hbar^2}{\mu}} \frac{1}{r}
\]

\(\ell = \sqrt{\frac{\hbar^2}{\mu}} \frac{1}{n^2}\)
Fig. 7-6. The radial probability distribution function $r^2 \chi^2_{n,l}$ for several values of the quantum numbers $n, l$. (From E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra*, Cambridge University Press, Cambridge, 1953, by permission.)
HATOM ENERGY LEVELS
AND EFFECTIVE POTENTIAL CURVES

ENERGY IN UNITS OF $e^2/2\alpha_0$

$\ell=0$

$\ell=1$

$\ell=2$

Centrifugal terms feel up the quantum well
ENERGY LEVELS FOR A PURE COULOMB WELL

Continuum

\begin{align*}
2s & \quad 2p \\
3s & \quad 3p \\
4s & \quad 4p \\
5s & \quad 5p \\
5d & \quad 5f \\
6g &
\end{align*}

ENERGY (above lowest level, in units of $\frac{e^2}{2}\alpha^2$)

\begin{align*}
1.0 & \\
0.5 & \\
0 &
\end{align*}
RADIAL PROBLEMS - 5

Isotropic Harmonic Oscillator

\[
\left( -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{1}{2} kr^2 + \frac{\hbar^2 \ell(\ell+1)}{2ur^2} \right) \chi = E\chi
\]

Use for distance unit \((\frac{\hbar}{\mu a})^{1/2}\) with \(\omega = (k/m)^{1/2}\)

Use for energy unit \(\frac{1}{2} \hbar \omega\)

Then

\[
\frac{d^2\chi}{dr^2} + \left( E - \frac{r^2}{2} + \frac{\ell(\ell+1)}{r^2} \right) \chi = 0
\]

with \(\chi = 0\) at \(r = 0\) and \(\chi \to 0\) at \(r \to \infty\).

Bound state energy levels given by

\[
E = 2n+3, \ n = 0, 1, 2, \ldots
\]

Again an unusual case because no dependence on \(\ell\)

Classical turning points:

\[
2n+3 - r_o^2 - \frac{\ell(\ell+1)}{r_o^2} = 0
\]

\[
r_o = \left[ (n + \frac{3}{2}) \pm \sqrt{(n + \frac{3}{2})^2 - \ell(\ell+1)} \right]^{1/2}
\]

and \(U(r_o, \ell) = 2n+3\)

Minimum in \(U(r, \ell)\):

\[
2r_m - \frac{2\ell(\ell+1)}{3r_m} = 0
\]

or

\[
r_m = \left[ \ell(\ell+1) \right]^{1/4}\ 	ext{and} \ U(r_m, \ell) = 2[\ell(\ell+1)]^{1/2}
\]
RADIAL PROBLEMS - 6

Sketch $U(r,\ell)$ functions and wavefunctions, making use of what we know already from solution of the isotropic oscillator problem in Cartesian coordinates:

Energy in units of $\hbar \omega$

<table>
<thead>
<tr>
<th>$\ell = 0$</th>
<th>$\ell = 1$</th>
<th>$\ell = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{0\text{ (inner)}} = 0$</td>
<td>$r_{0\text{ (inner)}} = \left[\left(\frac{n+\frac{3}{2}}{2}\right) - \left(\frac{n+\frac{3}{2}}{2}\right)^2 - \frac{1}{2}\right]^\frac{1}{2}$</td>
<td>$r_{0\text{ (inner)}} = \left[\left(\frac{n+\frac{3}{2}}{2}\right) + \sqrt{\left(\frac{n+\frac{3}{2}}{2}\right)^2 - 2}\right]^\frac{1}{2}$</td>
</tr>
<tr>
<td>$r_{0\text{ (outer)}} = (2n+3)^{\frac{1}{2}}$</td>
<td>$r_{0\text{ (outer)}} = \left[\left(\frac{n+\frac{3}{2}}{2}\right) + \sqrt{\left(\frac{n+\frac{3}{2}}{2}\right)^2 - 2}\right]^\frac{1}{2}$</td>
<td>$r_{0\text{ (outer)}} = \left[\left(\frac{n+\frac{3}{2}}{2}\right) + \sqrt{\left(\frac{n+\frac{3}{2}}{2}\right)^2 - 2}\right]^\frac{1}{2}$</td>
</tr>
</tbody>
</table>

inner/outer

$\frac{n}{3} = 0, 2, 4, \ldots$

$\frac{n}{3} = 1, 3, 5, \ldots$

$\frac{n}{3} = 2, 4, 6, \ldots$

$\frac{n}{3} = 0, 1, 2, \ldots$

$\frac{n}{3} = 0, 1, 2, \ldots$

$\frac{n}{3} = 0, 1, 2, \ldots$

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$\frac{n}{3} = 0, 1, 2, \ldots$

$\frac{n}{3} = 0, 1, 2, \ldots$
ENERGY LEVELS AND EFFECTIVE POTENTIAL CURVES FOR ISOVORETIC HARMONIC OSCILLATOR

ENERGY, in units of ħω

n=4  n=3  n=2  n=1  n=0

l=2  l=1  l=0

doesn't exist due to uncertainty principle
ENERGY LEVELS FOR AN ISOTROPIC
HARMONIC OSCILLATOR

ENERGY (above lowest level, in units of \( \hbar \omega \))

- Level 6: 5s, 5p (168), 5d
- Level 5: 4f, 4g
- Level 4: 3s, 3p (10), 3d
- Level 3: 2s, 2p (8), 2d
- Level 2: 1s
- Level 1: 0s

Levels are labeled with their corresponding \( l \) values:
- \( l = 0 \), \( l = 1 \), \( l = 2 \), \( l = 3 \)
RADIAL PROBLEMS - 7

Note the relationships between the 1-D problem and the $\ell = 0$ case of the 3-D problem:

\begin{align*}
\text{1-D Oscillator} & \quad \text{3-D Oscillator} \\
\text{\(n_x = 5\)} & \quad \text{\(n = 4\)} \\
\text{\(n_x = 4\)} & \quad \text{\(n = 2\)} \\
\text{\(n_x = 3\)} & \quad \text{\(n = 2\)} \\
\text{\(n_x = 2\)} & \quad \text{\(n = 0\)} \\
\text{\(n_x = 1\)} & \quad \text{(for \(n_x = 1, 3, 5, \ldots\))} \\
\text{\(n_x = 0\)} & \quad \text{(for \(n_x = 1, 3, 5, \ldots\))}
\end{align*}

We see that inserting an infinite wall at the mid-point of the 1-D potential eliminates the $n_x = 0, 2, 4, \ldots$ solutions, which all have maxima or minima there, whereas the $n_x = 1, 3, 5, \ldots$ solutions remain good since they have nodes at the midpoint. Hence the $x > 0$ portions of the latter become the solutions for the 3-D, $\ell = 0$ problem, with

\[ n = n_x - 1 = 0, 2, 4, \ldots \] (for $n_x = 1, 3, 5, \ldots$).

For the $\ell > 0$ cases of the 3-D problem, there is no simple relation to the 1-D problem. However, there are of course simple relations with the solution of the 3-D problem in Cartesian coordinates. These involve resolving the $(n_x, n_y, n_z)$ degeneracies into the $(n, \ell, m)$ states appropriate for spherical polar coordinates.

Thus, as in the Kramer's treatment of spherical harmonics, one readily finds the following correspondences:
### RADIAL PROBLEMS - 8

<table>
<thead>
<tr>
<th>Energy (units of $\frac{1}{2} \hbar \omega$)</th>
<th>Degeneracy</th>
<th>Cartesian States ($n_x, n_y, n_z$)</th>
<th>No. of Quanta $n$</th>
<th>Angular Momentum $\ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{3}{2}$</td>
<td>1</td>
<td>(000)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{5}{2}$</td>
<td>3</td>
<td>(100), (010), (001)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{7}{2}$</td>
<td>6</td>
<td>(200), (110), (201), (020), (101), (002), (011)</td>
<td>2</td>
<td>0-1 state, 2-5 states</td>
</tr>
<tr>
<td>$\frac{9}{2}$</td>
<td>10</td>
<td>(300), (210), (201), (030), (120), (012), (111), (003), (102), (021)</td>
<td>3</td>
<td>1-3 states, 3-7 states</td>
</tr>
</tbody>
</table>

etc.

Linear combinations of the degenerate cartesian solutions for a particular $n$ give solutions corresponding to certain $\ell$ values, as follows:

- $n = 5$, $\ell = 4f$
- $n = 4$, $\ell = 4d$
- $n = 3$, $\ell = 4f$
- $n = 2$, $\ell = 3d$
- $n = 1$, $\ell = 2p$
- $n = 0$, $\ell = 1s$

$s = 0$, $\ell = 2; 3, 4$

etc.

**Spherical Well, Infinitely Deep**

Consider $V(r) = 0$ for $r < a$

$= \infty$ for $r \geq a$.

Use for distance unit $a$

Use for energy unit $\frac{\hbar^2}{2mu}$
RADIAL PROBLEMS - 9

Then
\[ \frac{d^2 \chi}{dr^2} + \left[ E - \frac{\ell(\ell+1)}{r^2} \right] \chi = 0 \] for \(0 < r < 1\). Now require \( \chi = 0 \) both at \( r = 0 \) and \( r = 1 \). For \( \ell = 0 \) the solution is identical to the 1-D case.

Thus have
\[ \frac{d^2 \chi}{dr^2} + E \chi = 0 \quad \text{for} \quad \ell = 0 \]

\( \chi(r) = N \sin \sqrt{E} r \) (The \( \cos \sqrt{E} r \) term is absent since need \( \chi = 0 \) at \( r = 0 \).)

To make \( \chi = 0 \) at \( r = 1 \) requires
\[ \sqrt{E} = n \pi \quad \text{or} \quad E = n^2 \pi^2 \] in our reduced units \( n = 1, 2, 3, \ldots \)

For \( \ell > 0 \) the energy levels will depend on the value of \( \ell \) as well as \( n \).

Note that here we cannot compare with the cartesian problem because that dealt with a cubical well and we are considering a spherical one. It turns out that in fact there are no degeneracies among energy levels of different \( \ell \) in this problem.

Examine classical turning points:
\[ U(r, \ell) = \frac{\ell(\ell+1)}{r^2} \quad \text{for} \quad 0 < r < 1 \]
\[ = \infty \quad \text{for} \quad r > 1. \]

For \( \ell > 0 \), \( r_0 \) (outer) = 1 always, whereas for \( r_0 \) (inner) we have:
\[ E - \frac{\ell(\ell+1)}{r_0^2} = 0 \quad \text{or} \quad r_0 \) (inner) \( = \sqrt{E} \frac{\ell(\ell+1)}{\ell(\ell+1)} \]

For \( \ell > 0 \), minimum in \( U(r, \ell) \) occurs at \( r_m = 1 \) and \( U_m = \ell(\ell+1) \). Hence, find the following results:
As $\ell$ increases for a given $n$, the levels shift upwards because the spatial extent of the wavefunction is compressed.
Energy Levels and Effective Potential Curves for an Infinite Spherical Well

- $l = 0$
- $l = 1$
- $l = 2$
- $l = 3$

Energy in units of $\hbar^2 / 2\mu a^2$
ENERGY LEVELS FOR AN INFINITELY DEEP SPHERICAL WELL

Energy levels in units of \( \frac{\hbar^2}{2m} \):

- 1s: \( (\frac{1}{2}, 0) \) 1.86
- 2s: \( (\frac{1}{2}, 1) \) 3.47
- 2p: \( (\frac{1}{2}, 2) \) 5.17
- 3p: \( (\frac{1}{2}, 3) \) 6.87
- 4p: \( (\frac{1}{2}, 4) \) 8.57
- 5f: \( (\frac{1}{2}, 5) \) 10.27

Other terms:

- 6p
- 7f
- 8g
- 9h
- 10i
- 11j
- 12k
- 13l

Other states above 10 units of energy levels.
The mathematical treatment of H-atom, 3-D isotropic oscillator and spherical well problems is available in many texts - our aim in these notes is merely to emphasize qualitative features without getting involved in tedious details. In every case, by resolving the problem into a set of problems, one for each \( l \) value, the qualitative form of the wavefunctions, order of levels, etc., can be deduced with practically no calculation.

For reference, we list below the radial wavefunctions, \( \chi_{n\ell}(r) \) for the 3 problems considered here, omitting in each case a normalization factor \( N_{n\ell} \) to be determined such that \( \int_0^\infty \chi^2 4\pi r^2 dr = 1 \). Reduced units as defined above are used.

<table>
<thead>
<tr>
<th>Problem</th>
<th>General ( \chi_{n\ell} )</th>
<th>( \chi_{n\ell} ) as ( r \to 0 )</th>
<th>( \chi_{n\ell} ) as ( r \to \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>H-Atom</td>
<td>( r^{\ell+1} e^{-r/n} L_n^{2\ell+1}(\frac{2r}{n}) )</td>
<td>( r^{\ell+1} )</td>
<td>( r^n e^{-r/n} )</td>
</tr>
<tr>
<td>Isotropic Oscillator</td>
<td>( r^{\ell+1} e^{-r^2/2} L_{n-\ell}^{\ell+1}(\frac{r^2}{2}) )</td>
<td>( r^{\ell+1} )</td>
<td>( r^{n+1} e^{-r^2/2} )</td>
</tr>
<tr>
<td>Spherical Well</td>
<td>( r j_\ell(kr) )</td>
<td>( r^{\ell+1} )</td>
<td>-</td>
</tr>
</tbody>
</table>

\( k = \sqrt{E} \) determined by \( j_\ell(k) = 0 \)

\( L_\ell^\alpha(z) \) is an associated Laguerre Polynomial, \( j_\ell(z) \) a spherical Bessel function.
**H ATOM RADIAL WAVEFUNCTIONS**

\[ \chi_{nl}(r) = N e^{-r/n} r^{\ell+1} \frac{L_{\ell+1}^{2\ell+1}(2r)}{n^{\ell-1}} \]

**Table of \( NL_{n-\ell-1}^{2\ell+1}(2r/n) \) Including the normalization factor \( N \)**

<table>
<thead>
<tr>
<th>( \ell = 0 )</th>
<th>( \ell = 1 )</th>
<th>( \ell = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 1 )</td>
<td>2</td>
<td>-</td>
</tr>
<tr>
<td>( n = 2 )</td>
<td>( \sqrt{8} (2 - r) )</td>
<td>( \frac{1}{\sqrt{24}} )</td>
</tr>
<tr>
<td>( n = 3 )</td>
<td>( \frac{2}{81\sqrt{3}} (27 - 18r - 2r^2) )</td>
<td>( \frac{4}{81\sqrt{6}} (6 - r) )</td>
</tr>
</tbody>
</table>

At small \( r \), \( \chi_{nl} \) + const. \( r^{\ell+1} \). At large \( r \), \( \chi_{nl} \) + const. \( r^n e^{-r/n} \).

**HARMONIC OSCILLATOR WAVEFUNCTIONS**

\[ \chi_{nl}(r) = N e^{-r^2/2} r^{\ell+1} \frac{L_{\ell+1}^{1/2}(\ell+1/2)}{1/2}(r^2) \]

**Table of \( L_{\ell+1}^{1/2}(\ell+1/2)(r^2) \)**

<table>
<thead>
<tr>
<th>( \ell = 0 )</th>
<th>( \ell = 1 )</th>
<th>( \ell = 2 )</th>
<th>( \ell = 3 )</th>
<th>( \ell = 4 )</th>
<th>( \ell = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n=0 )</td>
<td>( L_{1/2}^{1/2} = 1 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( n=1 )</td>
<td>-</td>
<td>( L_{1/2}^{1/2} = 1 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( n=2 )</td>
<td>( L_{1/2}^{1/2} = 3/2 - r^2 )</td>
<td>-</td>
<td>( L_{1/2}^{1/2} = 1 )</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( n=3 )</td>
<td>-</td>
<td>( L_{1/2}^{1/2} = 5/2 - r^2 )</td>
<td>-</td>
<td>( L_{1/2}^{1/2} = 1 )</td>
<td>-</td>
</tr>
<tr>
<td>( n=4 )</td>
<td>( L_{1/2}^{1/2} = 15/8 - 5/2 r^2 + 1/2 r^4 )</td>
<td>-</td>
<td>( L_{1/2}^{1/2} = 1 )</td>
<td>-</td>
<td>( L_{1/2}^{1/2} = 1 )</td>
</tr>
<tr>
<td>( n=5 )</td>
<td>-</td>
<td>( L_{1/2}^{1/2} = 35/8 - 7/2 r^2 + 1/2 r^4 )</td>
<td>-</td>
<td>( L_{1/2}^{1/2} = 1 )</td>
<td>-</td>
</tr>
</tbody>
</table>

At small \( r \), \( \chi_{nl} \) + const. \( r^{\ell+1} \). At large \( r \), \( \chi_{nl} \) + const. \( r^n e^{-r^2/2} \)
SPHERICAL WELL WAVEFUNCTIONS

\[ \chi_{n\ell}(r) = N_r j_\ell(kr) \]

\[ k = \sqrt{E} \text{ determined such that } j_\ell(k) = 0 \]

Table of \( j_\ell(kr) = \sqrt{\frac{\pi}{2kr}} J_{\ell + \frac{1}{2}}(kr) \)

<table>
<thead>
<tr>
<th>( \ell = 0 )</th>
<th>( \ell = 1 )</th>
<th>( \ell = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\sin \text{ kr}}{\text{ kr}} )</td>
<td>( \frac{\sin \text{ kr} - \cos \text{ kr}}{\text{ kr}} )</td>
<td>( \left[ \frac{3}{(\text{ kr})^3} - \frac{1}{\text{ kr}} \right] \frac{\sin \text{ kr} - \frac{3}{(\text{ kr})^2} \cos \text{ kr}}{\text{ kr}} )</td>
</tr>
</tbody>
</table>

For \( kr \ll 1 \)

\[ j_\ell(kr) = \frac{(kr)^\ell}{1 \cdot 3 \cdot 5 \ldots (2\ell + 1)} \]

For \( kr \gg 1 \)

\[ j_\ell(kr) = \frac{1}{kr} \left[ \cos \text{ kr} - \frac{(\ell + 1)\pi}{2} \right] \]

CLASSICAL TURNING POINT FOR ISOTROPIC 3-D HARMONIC OSCILLATOR

\[ U = r^2 + \frac{\ell(\ell + 1)}{r^2} \]

\[ \frac{r_{\text{inner}}}{r_{\text{outer}}} \] (slide rule calculation)

<table>
<thead>
<tr>
<th>( \ell = 0 )</th>
<th>( \ell = 1 )</th>
<th>( \ell = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 0 )</td>
<td>( 0/1.732 )</td>
<td>( 1.00/1.414 )</td>
</tr>
<tr>
<td>1</td>
<td>( 0/2.24 )</td>
<td>( 0.669/2.145 )</td>
</tr>
<tr>
<td>2</td>
<td>( 0/2.65 )</td>
<td>( 0.545/2.60 )</td>
</tr>
<tr>
<td>3</td>
<td>( 0/3.00 )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( 0/3.33 )</td>
<td></td>
</tr>
</tbody>
</table>

\[ U_m = 0 \]

\[ r_m = 0 \]

\[ U_m = 2.93 \]

\[ r_m = 1.191 \]

\[ U_m = 4.91 \]

\[ r_m = 1.57 \]
Energy Levels for a particle in an infinitely deep spherical well, in units of $\frac{\hbar^2}{2\mu a^2}$

<table>
<thead>
<tr>
<th>n</th>
<th>l = 0</th>
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<th>l = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1s</td>
<td>9.87</td>
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<td></td>
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<tr>
<td>2s</td>
<td>39.4</td>
<td>19.9</td>
<td>3d</td>
</tr>
<tr>
<td>3s</td>
<td>88.6</td>
<td>58.9</td>
<td>4d</td>
</tr>
<tr>
<td>4s</td>
<td>157.7</td>
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</tbody>
</table>