As compared with the classical limit, the quantal result (for large \( \ell \)) has rapid oscillations and puts some intensity in the classically forbidden regions (although this decays very quickly).
QUASI-CLASSICAL CASE

WKB or JWKB Approximation

\[
\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + (E - V)\psi = 0 \quad \text{or} \quad \frac{d^2 \psi}{dx^2} + \frac{p^2}{\hbar^2} \psi = 0, \quad p = \sqrt{2m(E-V)}
\]

If \( V \) is constant, solutions are \( \psi = \text{constant} \cdot e^{\pm \frac{i}{\hbar} \int p(x) dx} \). For the case where \( \lambda \) is small compared with region in which \( V \) changes appreciably, try an approximation of the form

\[
\psi(x) = F(x) e^{\pm \frac{i}{\hbar} \int p(x) dx}
\]

where \( p(x) \) and \( F(x) \) are slowly varying functions of position. Substitution in Schrödinger Equation gives

\[
\frac{\hbar}{ip} \frac{d^2 F}{dx^2} \pm \left( 2 \frac{dF}{dx} + \frac{1}{p} \frac{dp}{dx} F \right) = 0
\]

Since we assume \( \lambda = \hbar / p \) is small and \( F \) varies slowly with \( x \), we omit the first term and have

\[
2 \frac{dF}{dx} + \frac{1}{p} \frac{dp}{dx} = \frac{d}{dx} \ln(F^2 p) \approx 0
\]

and hence

\[ F = \text{const} / p^{1/2} \]

This approximate form for \( \psi(x) \) can be obtained more formally by expanding an exponential form in powers of \( \hbar \). It turns out the first term leads to the classical result, the second to the "old quantum theory" (Bohr-Sommerfeld condition) result, and the higher terms to corrections which provide an asymptotic approach to the exact solution.

Thus, use

\[
\psi(x) = e^{\frac{i}{\hbar} S}
\]

with \( S = S_0 + \frac{\hbar}{i} S_1 + \left( \frac{\hbar^2}{i} \right) S_2 + \ldots \)

\[
\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + (E - V)\psi \Rightarrow \frac{1}{2m} \left[ \left( \frac{\partial S}{\partial x} \right)^2 - i\hbar \frac{\partial^2 S}{\partial x^2} \right] + (V - E) = 0
\]

Terms without \( \hbar \) give:
See also Park, p 89ff, who uses $\psi(x) = F(x)e^{iS(x)}$ where $F$ and $S$ are real in the classically allowed region,

For our case, $\psi(x) = F(x)e^{\pm \frac{i}{\hbar} \int^x p_o(x)dx}$

The lower limit can be any constant

$$\psi'(x) = \left(F' \pm \frac{ip_o}{\hbar} F\right) e^{\pm \frac{i}{\hbar} \int^x p_o dx}$$

$$\psi'' = \left[(F'' \pm \frac{2ip_o}{\hbar} F' \pm \frac{ip_o}{\hbar} F) \pm \left(\frac{ip_o}{\hbar}\right) \left(F' \pm \frac{ip_o}{\hbar} F\right)\right] e^{\pm \frac{i}{\hbar} \int^x p_o dx}$$

$$= \left[F'' \pm \frac{2ip_o}{\hbar} F' \pm \frac{ip_o^2}{\hbar^2} (F' \pm \frac{ip_o}{\hbar} F) - \frac{p_o^2}{\hbar^2} F\right] e^{\pm \frac{i}{\hbar} \int^x p_o dx}$$

LHS of Schrödinger Equation:

$$\psi'' + \frac{p_o}{\hbar^2} \psi = \left(F'' \pm \frac{2ip_o}{\hbar} F' \pm \frac{ip_o}{\hbar} F\right) e^{\pm \frac{i}{\hbar} \int p dx}$$

Neglect as amplitude slowly varying over wavelength

$$\cdot \cdot \cdot \frac{2i}{\hbar} \left(p_o F'' + \frac{1}{2} p_o F'\right) = 0$$

Factor out $p_o^{1/2}$, which $\neq 0$ if not at classical turning point.

$$p_o^{1/2} F'' + \frac{1}{2} p_o^{-1/2} p_o ' F = 0$$

$$\frac{d}{dx} (p_o^{1/2} F) = 0$$

$$p_o^{1/2} F = \text{const}$$

$$F = \text{const} p_o^{-1/2}$$

$$|\psi|^2 \sim \frac{1}{p_o} \sim \frac{1}{v_o}$$
QUASI-CLASSICAL - 2

(1) \[ \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 = E - V \]

Linear in \( \hbar \):

(2) \[ \frac{\partial S}{\partial x} + \frac{1}{2} \frac{\partial^2 S}{\partial x^2} = 0 \]

Quadratic in \( \hbar \):

\[ \frac{\partial S}{\partial x} + \frac{1}{2} \left( \frac{\partial S}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial^2 S}{\partial x^2} = 0, \text{ etc.} \]

Thus we find

\[ \frac{\partial S}{\partial x} = \pm \sqrt{2m(E-V)} \text{ and } S = \pm \int p \, dx, \quad p = \sqrt{2m(E-V)} \]

\[ \frac{\partial^2 S/\partial x^2}{\partial S/\partial x} = - \frac{1}{2} \frac{\partial}{\partial x} \ln \left( \frac{\partial S}{\partial x} \right) \]

\[ S_1 = - \frac{1}{2} \ln p \]

etc.

Then, if only \( S_0 \) and \( S_1 \) terms are retained,

\[ \psi_\pm(x) = e^{\pm i \frac{\pi}{\hbar} \int p \, dx} \quad \frac{1}{2} \ln p \quad e^{\pm \frac{i}{\hbar} \int p \, dx} \]

Potential Well Problems:

\( E = V(a) = V(b) \) locates classical turning points \( x = a \) and \( x = b \).

In the region of classically allowed motion, \( II \), we have \( E > V \), \( p \) real, and \( \psi \) will be some linear combination of \( \psi_+ \) and \( \psi_- \). We write it as

\[ \psi_{II} = C p^{-1/2} \sin \left( \frac{1}{\hbar} \int_a^x p \, dx + \phi \right). \]

The phase angle \( \phi \) specifies the relative proportion of \( \psi_+ \) and \( \psi_- \) and will be determined below by comparison of the approximate \( \psi_{II} \) with the exact solution near \( x = a \).
Figures from Powell and Craseman "Quantum Mechanics".

Fig. 5-22. The Airy function $Ai(z) = (1/\pi) \int_0^{\infty} \cos \left( \frac{s^3}{3} + sz \right) ds$.

Fig. 5-23. WKB approximation to the harmonic-oscillator wave function in the state $n = 4$. To the accuracy of the graph, the WKB wave function (heavy line) coincides with the exact wave function (broken line) in the interior of the well. Near the classical turning point $x_2 = 3$, the WKB approximation breaks down. The Airy function (light line) coincides with the exact wave function at $x_2$ and connects the WKB approximations in the classical and non-classical regions. At small and large $x$, the Airy function deviates from the exact wave function.
In the region of classically forbidden motion, I and III, we have \( E < V \), \( p = i|p| \) imaginary. Here the requirement that \( \psi \) remain finite as \( |x| \to \infty \) determines the choice of \( \psi_+ \) or \( \psi_- \). Thus, take

\[
\psi_\pm = \text{const.} |p|^{-1/2} e^{\mp \int |p| dx / h}.
\]

\[
\psi_I = A |p|^{-1/2} e^{\frac{1}{h} \int_a^x |p| dx}, \quad \psi_{III} = B |p|^{-1/2} e^{\frac{1}{h} \int_b^x |p| dx},
\]

or \( \psi_- \) for \( x < a \), or \( \psi_+ \) for \( b < x \).

Need to determine \( A, B, C \) and \( \phi \) to connect solutions. The WKB solutions fail near the turning points \( a \) or \( b \). However, close to \( a \) or \( b \) we can approximate the potential as linear and use the exact solutions (Airy functions) to interpolate between regions I and II and II and III.

Near \( x = a \), \( V(x) = E - \alpha(x-a) \Rightarrow \frac{d^2\psi}{dx^2} + \frac{2m\alpha}{\hbar^2} (x-a)\psi = 0 \)

Near \( x = b \), \( V(x) = E + \beta(x-b) \Rightarrow \frac{d^2\psi}{dx^2} - \frac{2m\beta}{\hbar^2} (x-b)\psi = 0 \)

Put

\[
z = -\left( \frac{2m\alpha}{\hbar^2} \right)^{1/3} (x-a) \quad \text{or} \quad z = \left( \frac{2m\beta}{\hbar^2} \right)^{1/3} (x-b)
\]

\[
= \left( \frac{2m\alpha}{\hbar^2} \right)^{1/3} (a-x)
\]

and then have in either case

\[
\frac{d^2\psi}{dz^2} - z\psi = 0.
\]

The solution which vanishes asymptotically for large positive \( z \) (\( z > 0 \) corresponds to either \( x < a \) or to \( x > b \)) is

\[
\text{Ai}(z) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos \left( \frac{s^3}{3} + sz \right) ds
\]

For large \( |z| \) this has the asymptotic forms
We consider values of \( x \) close enough to \( a \) or \( b \) that the linear approximation to \( V(x) \) is adequate but large enough that these asymptotic forms for \( \text{Ai}(z) \) can be used. Such values of \( x \) always exist if the motion is quasi-classical throughout (i.e. change in \( \lambda = \hbar / p \) small compared to itself).

Near \( x = a \) we have

\[
p = \sqrt{2m(E-V)} = \sqrt{2m[E - \xi + \alpha(x-a) - V_1]} = \sqrt{2m\alpha}(x-a)
\]

\[
p = (2m\alpha\hbar)^{1/3} (-z)^{1/2}
\]

For \( x < a \):

\[
\frac{1}{\hbar} \int_x^a p \, dx = \sqrt{2m\alpha} \int_x^a p \, dx = \sqrt{a-x} \, dx = \frac{2}{3} \sqrt{2m\alpha} (a-x)^{3/2} = \frac{2}{3} z^{3/2}
\]

For \( x > a \):

\[
\frac{1}{\hbar} \int_x^a p \, dx = \sqrt{2m\alpha} \int_x^a p \, dx = \sqrt{x-a} \, dx = \frac{2}{3} \sqrt{2m\alpha} (x-a)^{3/2} = \frac{2}{3} (-z)^{3/2}
\]

Hence we see that near \( x = a \),

\[
\psi_I(\text{exact}) \sim \frac{N}{2\sqrt{\pi}} e^{-\frac{2}{3} z^{3/2}} = \frac{(2m\alpha\hbar)^{1/6} N}{2\sqrt{\pi}} |p|^{-1/2} \frac{1}{\hbar} \int_a^x p \, dx
\]

\[
\psi_II(\text{exact}) \sim \frac{N}{\sqrt{\pi}(-z)^{1/4}} \sin \left[ \frac{2}{3} (-z)^{3/2} + \frac{\pi}{4} \right] = \frac{(2m\alpha\hbar)^{1/6} N}{\sqrt{\pi}} p^{-1/2} \sin \left[ \frac{1}{\hbar} \int_a^x p \, dx + \frac{\pi}{4} \right]
\]

Thus we find that \( \psi_{II} \) (WKB) will match \( \psi_{II} \) (exact) if we take \( \phi = \frac{\pi}{4} \) and also that \( \psi_I \) (WKB) will match \( \psi_{II} \) (WKB) and \( \psi_I \) (exact) if we take \( A = \frac{1}{2} C \).

Hence

\[
\psi_I = \frac{C}{2|p|^{1/2} e^{-\frac{1}{\hbar} \int_a x p \, dx}} \quad \psi_{II} = \frac{C}{p^{1/2}} \sin \left( \frac{1}{\hbar} \int_a^x p \, dx + \frac{\pi}{4} \right)
\]

approximation to left of \( a \) \quad \text{approximation to right of \( a \)}

\[
x < a \quad a < x
\]
Similar analysis near \( x = b \) shows

\[
\psi_{II} = \frac{C'}{p} \sin \left( \frac{1}{h} \int_{x}^{b} p\,dx + \frac{\pi}{4} \right) \leftrightarrow \psi_{III} = \frac{C'}{2|p|^{1/2}} e^{-\frac{1}{h} \int_{x}^{b} |p|\,dx}
\]

approx to left of \( b \)

\[
\psi_{II} \text{ approx to right of } b
\]

Since the two approximations to \( \psi_{II} \) must be the same (except perhaps for the constant factors \( C \) and \( C' \)), we have

\[
C \sin \left( \frac{1}{h} \int_{a}^{x} p\,dx + \frac{\pi}{4} \right) = C' \sin \left( \frac{1}{h} \int_{x}^{b} p\,dx + \frac{\pi}{4} \right)
\]

denote by \( \theta \)

\[
\text{denote by } \theta'
\]

In order that \( C \sin \theta = C' \sin \theta' \) be an identity in \( x \), the sum of the phases, \( \theta + \theta' \), which is a constant, must be an integer multiple of \( \pi \), or

\[
\theta + \theta' = \frac{1}{h} \int_{a}^{b} p\,dx + \frac{1}{2} \pi = (n+1)\pi, \quad n = 0, 1, 2, \ldots
\]

with \( C = (-1)^{n}C' \). Hence

\[
\frac{1}{h} \int_{a}^{b} p\,dx = (n + \frac{1}{2})\pi
\]

or

\[
J = \int_{a}^{b} p\,dx = (n + \frac{1}{2})h
\]

where \( J = \int_{a}^{b} p\,dx \) and \( 2\pi h = h \). It is interesting to note that if we had kept only the \( S_{0} \) term (rather than \( S_{0} \) and \( S_{1} \)) in carrying out the matching procedure, we would have found instead \( \int_{a}^{b} p\,dx = nh \)

The relation \( J(E) = (n + \frac{1}{2})h \) allows us to determine the discrete values of \( E \) which will yield suitable wavefunctions by merely computing the classical action integral. The corresponding WKB approximation to the wavefunction is

\[
\psi = \frac{C}{2|p|^{1/2}} e^{-\frac{1}{h} \int_{a}^{x} |p|\,dx} \leftrightarrow \frac{C}{p^{1/2}} \sin \left( \frac{1}{h} \int_{a}^{x} p\,dx + \frac{\pi}{4} \right) \leftrightarrow (-1)^{n}e^{-\frac{1}{h} \int_{x}^{b} |p|\,dx}
\]

\[
x < a \quad a < x < b \quad b < x
\]

The integer \( n \) = number of nodes of \( \psi \), since the phase \( \theta = \frac{1}{h} \int_{a}^{x} p\,dx + \frac{\pi}{4} \) increases from \( \frac{\pi}{4} \) at \( x = a \) to \( (n + \frac{3}{4})\pi \) at \( x = b \) and therefore the sine must vanish \( n \) times in this range (whereas outside \( a < x < b \), \( \psi \) decreases monotonically and has no zeros at a finite distance).

In general, we only expect the WKB approximation to be highly accurate when \( n \) is large. However, in some cases the exact \( E \) have the same functional dependence on \( n \) for small and large \( n \). In such cases (e.g. Coulomb
QUASI-CLASSICAL - 6

field, harmonic oscillator), the WKB quantization rule, although really applicable only for large $n$, gives the exact result for $E_n$.

In normalizing the WKB wavefunction, we can restrict the integration range to $a < x < b$, since at large $n$ $\psi$ falls very rapidly outside this range. Then

$$C^2 \int_a^b \frac{dx}{p} \sin^2 \frac{1}{\hbar} \left( \int_a^x \frac{p dx}{\hbar} + \frac{\pi}{4} \right) = 1.$$

In the quasi-classical domain, the argument of the sine is rapidly varying so we replace $\sin^2(\cdot)$ by its mean value of $1/2$ and obtain

$$\frac{1}{2} C^2 \int_a^b \frac{dx}{p} \geq 1.$$

In terms of the frequency $\omega = 2\pi/t_o$ of the classical periodic motion, $t_o = 2m \int_a^b \frac{dx}{p}$, we have

$$C = (2\omega m/\pi)^{1/2}.$$ Note $\omega$ varies with energy $E_n$.

Since $t_o = dJ(E)/dE$, have $t_o \approx \frac{J(E_{n+1}) - J(E_n)}{E_{n+1} - E_n} = \frac{\hbar}{E_{n+1} - E_n} \approx \frac{2\pi}{\omega_{n+1,n}}$.

Can also obtain convenient estimate of spacing of levels from

$$E_{n+1} - E_n = \frac{h}{J(E_{n+1}) - J(E_n)} \approx \frac{h}{dJ(E)/dE} \approx \frac{\hbar}{t_o}.$$

Penetration Through a Potential Barrier

In treating the potential well problem, we discarded the increasing real exponential terms in the nonclassical regions, in order to keep $\psi(x)$ finite at $\pm \infty$. For a potential barrier, the increasing exponentials must be retained, since the nonclassical region is of finite width. Hence we need to use the connection formulas derived via $\text{Bi}(z)$ as well as those via $\text{Ai}(z)$.

Suppose beam of particles is incident from the left. Thus in region I will have incident and reflected waves, in II will have decaying and increasing exponentials, and in III only a transmitted wave. We therefore "work backwards" from $\psi_{III}$ to find via the connection formulas the proper linear
OUTLINE OF POTENTIAL BARRIER PROBLEM

\[ \psi_{III} = \frac{A}{p^{1/2}} \left[ \cos \left( \frac{1}{\hbar} \int_a^x pdx + \frac{\pi}{4} \right) + i \sin \left( \frac{1}{\hbar} \int_a^x pdx + \frac{\pi}{4} \right) \right], \quad x > a \]

\[ \psi_{II} = \frac{A}{|p|^{1/2}} \left[ e^{-\frac{1}{\hbar} \int_b^x |p|dx} + \frac{i}{2} T e^{\frac{1}{\hbar} \int_b^x |p|dx} \right], \quad b < x < a \]

\[ \psi_I = \frac{A}{p^{1/2}} \left[ 2T^{-1} \sin \left( \frac{1}{\hbar} \int_x^b pdx + \frac{\pi}{4} \right) + \frac{i}{2} T \cos \left( \frac{1}{\hbar} \int_x^b pdx + \frac{\pi}{4} \right) \right], \quad x < b \]
QUASI-CLASSICAL - 7

Combinations to represent $\psi_{II}$ and $\psi_{I}$. Form of $\psi_{III}$ is

$$\psi_{III} = \frac{A}{p^{1/2}} e^{i \left( \frac{1}{\hbar} \int_{a}^{X} p\,dx + \frac{\pi}{4} \right)}, \ x > a$$

where we have inserted the $\frac{\pi}{4}$ in the exponential to facilitate application of the connection formulas. Since $A$ is complex such a phase factor may be absorbed in it. To apply the connection formulas, we first write

$$\psi_{III} = \frac{A}{p^{1/2}} \left[ \cos \left( \frac{1}{\hbar} \int_{a}^{X} p\,dx + \frac{\pi}{4} \right) + i \sin \left( \frac{1}{\hbar} \int_{a}^{X} p\,dx + \frac{\pi}{4} \right) \right], \ x > a$$

Then, using the formulas for a "barrier at the left," we find in the non-classical region

$$\psi_{II} = \frac{A}{|p|^{1/2}} \left[ e^{-\frac{1}{\hbar} \int_{x}^{a} |p|\,dx} + \frac{i}{2} e^{-\frac{1}{\hbar} \int_{x}^{a} |p|\,dx} \right], \ x < a$$

Now we rewrite $\psi_{II}$ in a form convenient for deriving $\psi_{I}$ by use of connection formulas for the "barrier to right" case, which involve $\int_{b}^{X}$ or $\int_{X}^{b}$. Thus, use

$$\int_{x}^{a} |p|\,dx = \int_{b}^{a} |p|\,dx - \int_{b}^{x} |p|\,dx.$$  

Define

$$T = e^{-\frac{1}{\hbar} \int_{b}^{a} |p|\,dx}$$

and hence rewrite $\psi_{II}$ as

$$\psi_{II} = \frac{A}{p^{1/2}} \left[ T^{-1} e^{-\frac{1}{\hbar} \int_{b}^{X} |p|\,dx} + \frac{i}{2} T e^{-\frac{1}{\hbar} \int_{b}^{X} |p|\,dx} \right], \ b < x < a.$$  

Now obtain via the connection formulas

$$\psi_{I} = \frac{A}{p^{1/2}} \left[ 2T^{-1} \sin \left( \frac{1}{\hbar} \int_{X}^{b} p\,dx + \frac{\pi}{4} \right) + \frac{i}{2} T \cos \left( \frac{1}{\hbar} \int_{X}^{b} p\,dx + \frac{\pi}{4} \right) \right], \ x < h.$$
Finally, in order to identify the incident and reflected parts of \( \psi_1 \), it is convenient to rewrite it in terms of imaginary exponentials,

\[
\psi_1 = \frac{A}{p_1^{1/2}} \left[ 2T^{-1} \left( \frac{e^{i\theta} - e^{-i\theta}}{2i} \right) + \frac{1}{2} T \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right) \right] , \quad \theta = \frac{1}{R} \int_x^b p \, dx + \frac{\pi}{4}
\]

\[
= \frac{-iA}{p_1^{1/2}} [(T^{-1} - \frac{1}{4} T)e^{i\theta} - (T^{-1} + \frac{1}{4} T)e^{-i\theta}] \equiv \psi_1^+ + \psi_1^-
\]

Now see that the \( \psi_1^+ \sim e^{i\theta} \) term represents a wave moving to the left, hence the reflected wave, and the \( \psi_1^- \sim e^{-i\theta} \) term represents a wave moving to the right, the incident wave. Amplitudes of interest are:

Incoming wave \( \psi_1^- : \frac{|A|}{p_1^{1/2}} (T^{-1} + \frac{1}{4} T) \equiv a_1^- \)

Reflected wave \( \psi_1^+ : \frac{|A|}{p_1^{1/2}} (T^{-1} - \frac{1}{4} T) \equiv a_1^+ \)

Transmitted wave \( \psi_{III} : \frac{|A|}{p_{III}^{1/2}} a_{III} \)

The transmission coefficient is defined as ratio of transmitted to incident flux, where flux is given by velocity times intensity, or \( v|a|^2 \), and thus

\[
\text{Transmission coefficient} = \mathcal{T} = \frac{v_{III} |a_{III}|^2}{v_{I} |a_I^-|^2} = \frac{p_{III} |a_{III}|^2}{p_{I} |a_I^-|^2}
\]

\[
\mathcal{T} = \frac{T^2}{(1 + \frac{1}{4} T^2)^2} = T^2 e^{-\frac{2}{R} \int_a^b \sqrt{2m(V-E)} \, dx}
\]

as \( T \) must be small for WKB treatment to be valid.

Similarly,

\[
\text{Reflection coefficient} = \mathcal{R} = \frac{v_{I} |a_I^-|^2}{v_{I} |a_I^+|^2} = \frac{|a_I^-|^2}{|a_I^+|^2}
\]
Solutions of the differential equation
\[ \frac{d^2\psi}{dz^2} - z\psi = 0 \]
are called Airy functions. The solution which vanishes for large positive \( z \) is
\[ Ai(z) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{s^3}{3} + sz\right) ds \]
with the following asymptotic forms for large \( |z| \),
\[ Ai(z) \to \frac{1}{2\sqrt{\pi} z^{1/4}} e^{-\frac{2}{3} z^{3/2}}, \quad z > 0 \]
\[ Ai(z) \to \frac{1}{\sqrt{\pi}(-z)^{1/4}} \sin\left[\frac{2}{3} (-z)^{3/2} + \frac{\pi}{4}\right], \quad z < 0 \]
A second solution, which diverges for large positive \( z \), is
\[ Bi(z) = \frac{1}{\pi} \int_0^\infty \left[ e^{-sz} - \frac{1}{3} s^3 + \sin\left(\frac{s^3}{3} + sz\right) \right] ds \]
with the asymptotic forms
\[ Bi(z) \to \frac{1}{\sqrt{\pi} z^{1/4}} e^{\frac{2}{3} z^{3/2}}, \quad z > 0 \]
\[ Bi(z) \to \frac{1}{\sqrt{\pi}(-z)^{1/4}} \cos\left[\frac{2}{3} (-z)^{3/2} + \frac{\pi}{4}\right], \quad z < 0 \]
Note the factor of \( \frac{1}{2} \) that appears in the exponential forms for \( Ai(z) \) and \( Bi(z) \).
SUMMARY OF CONNECTION FORMULAS

"Barrier to left"

\[ V(x) = E \alpha (x-a) + \ldots \]

\[ V(x) = E \]

\[ \frac{1}{2|p|^{1/2}} \int_a^x |p| \, dx + \frac{1}{p^{1/2}} \sin \left( \frac{1}{h} \int_a^x pdx + \frac{\pi}{4} \right) \]

\[ \frac{1}{h} \int_a^x |p| \, dx \to Ai(z) + \frac{1}{p^{1/2}} \sin \left( \frac{1}{h} \int_a^x pdx + \frac{\pi}{4} \right) \]

\[ \frac{1}{h} \int_a^x |p| \, dx + Bi(z) + \frac{1}{p^{1/2}} \cos \left( \frac{1}{h} \int_a^x pdx + \frac{\pi}{4} \right) \]

\[ \frac{1}{h} \int_a^x |p| \, dx = \frac{2}{3} z^{3/2} \]

\[ z = \frac{-p^2}{(2mc\hbar)^{2/3}} \]

\[ \frac{1}{h} \int_a^x pdx = \frac{2}{3} (-z)^{3/2} \]

"Barrier to right"

\[ V(x) = E + \beta (x-b) + \ldots \]

\[ V(x) = E \]

\[ \frac{1}{p^{1/2}} \sin \left( \frac{1}{h} \int_a^x pdx + \frac{\pi}{4} \right) + Ai(z) + \frac{1}{2|p|^{1/2}} e^{-\frac{1}{h} \int_b^x |p| \, dx} \]
SUMMARY CONNECTION FORMULAS - 2

\[ \frac{1}{p^{1/2}} \cos \left( \frac{1}{\hbar} \int_x^b p\,dx + \frac{\pi}{4} \right) \rightarrow \text{Bi}(z) + \frac{1}{|p|^{1/2}} e^{\frac{1}{\hbar} \int_b^x |p|\,dx} \]

\[ \frac{1}{\hbar} \int_x^b p\,dx = \frac{2}{3} (-z)^{3/2} \]

\[ z = \frac{-p^2}{(2\pi m \hbar)^{2/3}} \]

\[ \frac{1}{\hbar} \int_b^x |p|\,dx = \frac{2}{3} z^{3/2} \]