Asymptotic Approximation for Legendre Polynomials (Landau & Lifschitz, p. 166-168)

\[
\left[ \frac{d^2}{d\theta^2} + \cot\theta \frac{d}{d\theta} + \ell(\ell+1) \right] P_\ell(\cos\theta) = 0.
\]

Substitute

\[
P_\ell(\cos\theta) = \frac{x(\theta)}{\sqrt{\sin\theta}}
\]

Find

\[
\frac{d^2x}{d\theta^2} + \left[ (\ell + \frac{1}{2})^2 + \frac{1}{4} \csc^2\theta \right] x = 0.
\]

This has the form of a one-dimensional wave equation,

\[
x'' + \lambda^{-2}x = 0
\]

where

\[
\lambda = \left[ (\ell + \frac{1}{2})^2 + \frac{1}{4} \csc^2\theta \right]^{-\frac{1}{2}} = \frac{1}{\left( \ell + \frac{1}{2} \right)^2 + \frac{1}{4} \sin^2\theta}
\]

has the role of a deBroglie wavelength. In order to apply the quasiclassical approximation, we require \(d\lambda/d\theta\) to be small, or \(|(\ell + \frac{1}{2})\sin\theta|\) large. For large \(\ell\), this condition will hold except near \(\theta = 0\) and \(\theta = \pi\), where \(\sin\theta\) vanishes. Thus we must require \(\ell \xi >> 1\) and \((\pi - \theta) \xi >> 1\). When these conditions hold, solve

\[
x'' + \left( \ell + \frac{1}{2} \right)^2 x = 0
\]

and obtain

\[
x(\theta) = A \sin\left( \left( \ell + \frac{1}{2} \right) \theta + \alpha \right)
\]

or

\[
P_\ell(\cos\theta) = A \frac{\sin\left( \left( \ell + \frac{1}{2} \right) \theta + \alpha \right)}{\sqrt{\sin\theta}}, \text{ for } \theta >> \frac{1}{\xi}, (\pi - \theta) >> \frac{1}{\xi}
\]

To determine the constants \(A\) and \(\alpha\), proceed as usual in the WKB method by comparing with result of an exact solution in the region where the quasiclassical approximation fails.

When \(\theta\) is small enough, the differential equation for \(P_\ell(\cos\theta)\) can be solved in terms of the Bessel function of zero order, \(J_0(x)\). Thus, for
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$\theta \ll 1$, put $\cot \theta = 1/\theta$ and replace $\ell (\ell + 1)$ by \((\ell + \frac{1}{2})^2\), to obtain

$$\left[ \frac{d^2}{d\theta^2} + \frac{1}{\theta} \frac{d}{d\theta} + \left( \ell + \frac{1}{2} \right)^2 \right] p_{\ell}(\cos \theta) = 0.$$  

This is the Bessel equation, with the solution

$$p_{\ell}(\cos \theta) = J_\ell \left( \left( \ell + \frac{1}{2} \right) \theta \right), \quad \ell \theta \ll 1.$$  

This approximation holds at $\theta = 0$ and at larger angles as long as $\theta \ll 1$. In particular it can be applied in the range $1/\ell \ll \theta \ll 1$, where it must agree with our previous result. The asymptotic expansion of the Bessel function is

$$J_\ell(x) = \sqrt{\frac{2}{\pi x}} \sin \left( x + \frac{\pi}{4} \right) \quad \text{for} \ x \gg 1.$$  

Thus, with $x = \left( \ell + \frac{1}{2} \right) \theta$ we have

$$p_{\ell}(\cos \theta) = \left( \frac{2}{\pi (\ell + \frac{1}{2}) \theta} \right)^{\frac{1}{2}} \sin \left( (\ell + \frac{1}{2}) \theta + \frac{\pi}{4} \right) \quad \text{for} \ 1/\ell \ll \theta \ll 1.$$  

So we find

$$A = \sqrt{\frac{2}{\pi (\ell + \frac{1}{2})}} \quad \text{and} \quad \alpha = \frac{\pi}{4} \quad \text{and hence}$$

$$p_{\ell}(\cos \theta) = \sqrt{\frac{2}{\pi (\ell + \frac{1}{2})}} \frac{\sin \left( (\ell + \frac{1}{2}) \theta + \frac{\pi}{4} \right)}{\sqrt{\sin \theta}} \quad \text{for} \ 1/\ell \ll \theta$$

$$= J_0 \left[ \left( \ell + \frac{1}{2} \right) \sin \theta \right] \quad \text{for} \ell \theta \ll 1 \text{ or } (\pi - \theta) \ll 1.$$  

Note that the normalized angular wave function $\Phi_{\ell m}(\theta)$ for $m = 0$ is

$$\Phi_{\ell 0}(\theta) \propto \sqrt{\frac{2\ell + 1}{2}} \ p_{\ell}(\cos \theta).$$  

Thus

$$|\Phi_{\ell 0}|^2 = \frac{2}{\pi} \frac{\sin^2 \left[ (\ell + \frac{1}{2}) \theta + \frac{\pi}{4} \right]}{\sin \theta} \quad \text{for} \ \theta \gg \frac{1}{\ell}, \ (\pi - \theta) \gg \frac{1}{\ell}. $$
For large $\ell$, the $\sin^2$ factor in the numerator oscillates very rapidly and can be replaced by its average value, $1/2$. Then

$$|\Theta_{20}|^2 = \frac{1}{\pi \sin \theta},$$

for large $\ell$ holds everywhere except very close to $\theta = 0$ and $\pi$.

Physical Interpretation:

For large $\ell$, the quasiclassical motion corresponds to particle rotating about the angular momentum vector, $\mathbf{J}$. For $m = 0$, the $\mathbf{J}$ vector is perpendicular to the $z$-axis, so the situation is:

If $\theta$ is the azimuthal angle about $\mathbf{J}$, the probability of finding the particle in $\theta$ to $\theta + d\theta$ is uniform. But $\mathbf{J}$ may have any azimuthal orientation about $z$, i.e., plane in which the particle moves may have any $\phi$ with equal probability.

To obtain the full distribution, rotate about $z$-axis: Note that the probability contained in the interval $\theta$ to $\theta + d\theta$ for the particular plane pictured on the extreme left is spread out over $2\pi \sin \theta d\theta$ when averaged over $\phi$, hence the probability in the full distribution becomes proportional to

$$\frac{1}{\sin \theta}$$

Another way to visualize this result is to consider the density of intersections of latitude and longitude lines on a sphere. Clearly, these intersections crowd together near the poles and, from the solid angle element, the density of intersections is seen to be inversely proportional to $\sin \theta$.

The situation for $m \neq 0$ can be considered in the same fashion. Suppose projection of $\mathbf{J}$ on the $z$-axis is $M$, then the quasiclassical motion corresponds to:

where $\cos \alpha = M/L$. Again, for any particular orientation of the $\mathbf{J}$ vector, the probability of finding the particle is uniform in the azimuthal angle about $\mathbf{J}$, and $\mathbf{J}$ may have any azimuthal orientation about $z$, but with $\alpha$ fixed.
If we let $\psi$ be the angle of rotation in the plane perpendicular to $\lambda$, then according to the argument in the center of page 3, we have

$$P_\theta(\theta)\sin\theta d\theta = NP_\psi(\psi) d\psi$$

where $P_\theta(\theta)$ refers to the probability of finding the particle between $\theta$ and $\theta + d\theta$ and $P_\psi(\psi)$ to that of finding it in $\psi$ to $\psi + d\psi$, and $N$ is a normalization constant. Since $P_\psi(\psi) = \text{constant}$, we have

$$P_\theta(\theta) = \frac{N(d\psi/d\theta)}{\sin\theta}.$$ 

In the $m = 0$ case $\psi = \theta$ and the previous result follows. In the $m \neq 0$ case, we must determine the relation between $\psi$, $\theta$, and $\alpha$.

Note $\psi$ is dihedral angle between planes containing $(\lambda_0$ and $\lambda)$ and $(\lambda_0$ and $\lambda)$. Hence we use the spherical trigonometry formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

Here $a = \theta$, $b = \alpha$, $c = \frac{\pi}{2}$, $A = \psi$.

Thus we have

$$\cos \theta = \sin \alpha \cos \psi$$

$$\sin \theta d\theta = \sin \alpha \sin \psi d\psi$$

so

$$P_\theta(\theta) = \frac{N}{\sin \alpha \sin \psi} = \frac{N}{(\sin^2 \alpha - \cos^2 \theta)^{1/2}} = \frac{N}{(\sin^2 \theta - \cos^2 \alpha)^{1/2}}$$

Note that the region where $\sin^2 \theta < \cos^2 \alpha$, corresponding to either $\theta < \frac{\pi}{2} - \alpha$ or $\theta > \frac{\pi}{2} + \alpha$, is classically forbidden.
As compared with the classical limit, the quantal result (for large $\ell$) has rapid oscillations and puts some intensity in the classically forbidden regions (although this decays very quickly).