Consider scattering from an attractive spherical potential well of depth $V_0$ and radius $a$, given by $V(r) = -V_0$ for $r < a$ and $V(r) = 0$ for $r > a$. Suppose the collision energy is high enough to make the deBroglie wavelength small compared with the well radius, so $ka << 1$. Thus you need deal only with the $l = 0$ partial wave.

(i) Derive from the radial Schrödinger equation solutions for $u_0(r) = rR_0(r)$ inside and outside the well: show that for

$$r < a, \quad d^2u/dr^2 + k^2u = 0 \quad \text{with} \quad k = (2\mu V_0/k^2)^{1/2}$$

when $E \to 0$ (i.e., $E << V(r)$)

and for $r > a, \quad d^2u/dr^2 + k^2u = 0 \quad \text{with} \quad k = (2\mu E/k^2)^{1/2}$

(since here $E >> V(r)$). From these show that the solutions satisfying boundary conditions are

$$u_{in} = Asinkr \quad \text{and} \quad u_{out} = Bsin(kr + \eta_0)$$

(ii) Equate logarithmic derivatives of $u_{in}$ and $u_{out}$ at $r = a$ to obtain

$$k \cot ka = k \cot (ka + \eta_0) \equiv k/(ka + \eta_0)$$

and thereby evaluate $\eta_0$. Derive expressions for the total cross section $\sigma$ and graph $\sigma/4\pi a^2$ versus $ka$. Note that if the sign of $V_0$ is changed (to produce a repulsive "step" potential instead of an attractive well), the corresponding $\sigma$ can be obtained by $k \to ik'$. Plot that $\sigma/4\pi a^2$ likewise versus $k'a$. Also $\sigma$ in the limit $V_0 \to \infty$.

The potential energy for scattering of an electron by an atom can be represented approximately as a "shielded" Coulomb field by the "Yukawa potential" given by

$$V(r) = (Ze^2/r)e^{-r/a}$$

where $a$ is a "shielding radius." Show that in the Born approximation the scattering amplitude is

$$f(\theta) = \frac{2\mu Ze^2}{k^2 K^2} \left[ \frac{1}{1 + (Ka)^2} \right]$$
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Problem #3

5. Consider a Lennard-Jones \((n, m)\) potential,

\[ V(r) = C_m r^{-m} + C_n r^{-n} \]

where \(1 < m < n\). The constants are related to the depth \(\varepsilon\) and radius \(r_m\) of the well by

\[ C_m = \frac{n}{n(n-m)} \varepsilon r_m^n, \quad C_n = \frac{m}{n(n-m)} \varepsilon r_m^n \]

a) Starting with the Jeffrey-Born approximation for the phase shift, derive expressions for \(\eta(b)\), \(\chi(b)\), \(\chi'(b)\) and \(\chi''(b)\) in the high energy limit. Use the customary reduced variables: \(\beta = b/r_m\), \(D = 2\varepsilon r_m / \hbar v\), \(K = E/\varepsilon\), etc.

b) From your results derive formulas for the high energy limits of \(\beta_g\) and \(\beta_r\) as functions of the potential shape parameters \(m\) and \(n\). Evaluate these for one of the 5 cases: \(m = 4\), \(n = 8, 12, 20\); \(m = 6\) and \(n = 8, 20\). Use these results to determine the corresponding \(\eta_g\), \(a_g\), \(\eta_r\), \(\theta_r\), \(a_r\) parameters for that potential in terms of the reduced variables. Note the variation with \(m\) and \(n\) by comparing with the \(n = 6, 12\) results given in class notes and with results for other potentials treated by your comrades. [Note: Use \(\chi = \chi_g + a_g (1 - \chi_g)\)]

c) Derive the value of the critical reduced energy \(K_c\) for orbiting for the LJ\((n,m)\) potential. Use the fact that \(\theta_r \rightarrow \infty\) is \(K \rightarrow K_c\) from above and the high energy expression for \(\theta_r\) (which becomes accurate for \(K \gtrsim 3\) or 4) to sketch the dependence of \(\chi_r\) on \(K\) for the potential you selected in (b). Again compare with other potentials.

d) Evaluate the Landau-Lifschitz approximation for the total cross section, \(Q_{LL}\), for your potential, in the low \((m\) term) and high \((n\) term) velocity regions. Likewise evaluate the glory contribution, \(\Delta Q_g\). For comparison, the results for the LJ\((6,12)\) case are:

\[ \frac{Q_{LL}}{\pi r_m^2} = 2.57 \ D^{2/5} \]
\[ = 1.85 \ D^{2/11} \]
\[ \Delta Q_g / \pi r_m^2 \equiv 1.32 \ D^{-1/2} \cos 2\pi N_g \]

Using the high energy \(\beta_g, a_g\) parameters.
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e) Consider a system with $B = 500$ subject to your potential. Plot the rainbow pattern, in the Ford-Wheeler approximation, for $K = 5$. Also, plot the glory undulations. For simplicity, use the high energy approximation for all the parameters.

6. Use the results obtained with the high energy approximation for a LJ(6,12) with $B = 500$ to examine the relation between supernumery rainbows and glory undulations. First show that the classical rainbow angle $\chi_l$ becomes equal to or larger than the "Heisenberg angle" $\chi_H = \lambda/b = a/(A\beta)$ for $D \geq 2.12$. Then derive from the Airy function the values of $D$ which correspond to the maxima and minima of the rainbow structure. Compare these with the location of extrema in the glory undulations.