19. Inverse Power Potentials

For the inverse power potentials

\[ V(r) = \frac{-C}{r^s}, \text{ attractive, } C > 0, \ s \geq 1 \]

\[ V(r) = \frac{C'}{r^s}, \text{ repulsive, } C' > 0, \ s \geq 1 \]

the classical angle of deflection can be evaluated in terms of trigonometric functions when \( s = 1 \) or \( 2 \) and in terms of elliptic integrals when \( s = 4 \) or \( 6 \) and in several other cases.

We shall consider first the cases \( s = 1 \) and \( 2 \). For both of these, the potential approaches zero so slowly at large \( r \) that the phase shift and collision lifetime are infinite; also \( I(0) \) and \( \sigma \) are infinite. In both cases, the results for a repulsive potential are very simply related to those for an attractive potential.

For the \( s = 1 \) case, which would correspond to gravitational or coulombic attraction, \( V(r) = -C/r \), and

\[ \chi = \pi - 2b \int_{r_c}^{\infty} \frac{dr}{r^2 \sqrt{1 - \frac{b^2}{r^2} + \frac{C}{Er_c}}} \overset{\text{(19-1)}}{=} \frac{1}{\sqrt{r_c}} \left[ 1 - \frac{b^2}{r_c^2} + \frac{C}{Er_c} \right]^{1/2} \]

The turning point is defined by

\[ 1 - \frac{b^2}{r_c^2} + \frac{C}{Er_c} = 0 \]

or

\[ r_c^2 + \frac{C}{E} r_c - b^2 = 0 \]
\[ r_c = \frac{C}{2b} \left[ 1 + \left( \frac{C}{b} \right)^2 + 4b^2 \right]^{1/2} \]  

(19-3)

The substitution \( x = r_c/r \) converts (19-1) to

\[
\chi = \pi - 2 \left[ \frac{1}{r}\int_0^1 \frac{dx}{\sqrt{r_c^2 - x^2 + \frac{Cr_c x}{Eb} 1/2}} \right]
\]

or, using (19-2),

\[
\chi = \pi - 2 \left[ \frac{1}{r}\int_0^1 \frac{dx}{\sqrt{r_c^2 - x^2 + \left(1 - \frac{r_c^2}{b^2}\right)x}} \right]^{1/2}
\]

The integration yields

\[
\chi = -2 \left[ \arcsin \left( \frac{1 - \frac{r_c}{b}}{1 + \frac{r_c}{b}} \right) \right]^{1/2}
\]

From Fig. 19-1, we see that this may be rewritten as

\[
\chi = -2 \arctan \left( \frac{b^2 - r_c^2}{2br_c} \right)
\]

and (19-2) gives \( b^2 - r_c^2 = \frac{C}{E}r_c \), so we write the final result as

\[
\chi = -2 \arctan \left( \frac{\frac{C}{2Eb}}{2br_c} \right) \quad (19a-4)
\]
For the case of a repulsive potential, \( V(r) = C'/r \), the calculation is the same; however Eq. (19-3) becomes

\[
    r_c' = \frac{C'}{2E} + \frac{1}{2} \left[ \frac{C'}{2E} \right]^2 + 4b' \left( \frac{1}{2} \right)^{1/2}
\]

(19-5)

and (19-4) becomes

\[
    \chi' = 2 \arctan \left( \frac{C'}{2Eb'} \right)
\]

(19-6)

Fig. 19-2 shows a typical hyperbolic trajectory which would obtain for the case of an attractive center at A or for a repulsive center at B (with \( C' = C \)). Note that \( \chi'(b') = 2\pi - \chi(b) \). The points A and B are at the interior and exterior foci of the hyperbolic trajectory.

Fig. 19-2

The differential cross section is evaluated from

\[
    I(\chi) = \frac{b}{|\sin \chi d\chi/db|}
\]

From (19-4),

\[
    b = \frac{C}{2E} \cot \frac{\chi}{2}
\]

and

\[
    \frac{d\chi}{db} = -2 \left[ 1 + \frac{C}{2Eb} \right]^{-1} \left( -\frac{C}{2Eb} \right)
\]

\[
    = \frac{4E}{C} \left[ 1 + \tan^2 \frac{\chi}{2} \right]^{-1} \tan \frac{\chi}{2} = \frac{4E}{C} \sin^2 \frac{\chi}{2}
\]
Thus

\[ I(\chi) = \frac{1}{\pi} \left( \frac{C}{2E} \right)^2 \cot^2 \frac{\chi}{2} \frac{\cos^2 \frac{\chi}{2}}{\sin \chi \sin^2 \frac{\chi}{2}} = \frac{1}{4} \left( \frac{C}{2E} \right)^2 \frac{\cos^2 \frac{\chi}{2}}{\cos^2 \frac{\chi}{2} \sin^2 \frac{\chi}{2}} \]

\[ I(\chi) = \left( \frac{C}{E} \right)^2 \left( \frac{1}{2 \sin^2 \frac{\chi}{2}} \right)^4 \quad (19-7) \]

For the scattering of ions or charges \( Z_1 e \) and \( Z_2 e \), \( C = Z_1 Z_2 e^2 \) and the familiar Rutherford formula is obtained. Fig. 19-3 illustrates the extremely steep decrease with \( \chi \) which arises because of the very long range interaction for the \( s = 1 \) case.

For the \( s = 2 \) case, the calculation is even simpler. With \( V(r) = -C/r^2 \),

\[ \chi = \pi - 2b \int_0^\infty \frac{dr}{r^2 \left[ 1 - \frac{b^2}{r^2} + \frac{C}{E} \frac{1}{r^2} \right]^{1/2}} \quad (19-8) \]

The turning point is determined from

\[ b^2 = r_c^2 \left[ 1 + \frac{C}{E} \frac{1}{r_c^2} \right] \quad (19-9) \]

or

\[ r_c = \left[ b^2 - \frac{C}{E} \right]^{1/2} \]

The substitution \( x = r_c/r \) converts (19-8) into

\[ \chi = \pi - \frac{2b}{r_c} \int_0^\infty \frac{dx}{\left[ 1 - \left( \frac{b^2}{r_c^2} - \frac{C}{E} \frac{1}{r_c^2} \right) x^2 \right]^{1/2}} \]

and, from (19-9) we see that the coefficient of \( x^2 \) in the denominator is unity. Thus
\[ \chi = \pi - \frac{2b}{r_c} \arcsin x \bigg|_0^1 = \pi \left(1 - \frac{b}{r_c}\right) \]

or

\[ \chi = \pi \left(1 - \frac{b}{(b^2 - C/E)^{1/2}}\right) \] (19-10)

The same result obtains for a repulsive potential, with C replaced by \(-C'\).

Since

\[ \frac{\pi - \chi}{\pi} = \frac{b}{r_c} \]

or

\[ b^2 = \frac{C}{E} \left(\frac{\pi - \chi}{\pi}\right)^2 \left[1 - \left(\frac{\pi - \chi}{\pi}\right)^2\right]. \]

the differential cross section is found to be

\[ I(\chi) = \frac{|C/E|}{\sin \chi} \left(\frac{\pi - \chi}{\pi}\right) \left[1 - \left(\frac{\pi - \chi}{\pi}\right)^2\right]^{-2} \] (19-11)

This is also sketched in Fig. 19-3.