Homework: Why $l(l+1)$?

Investigate the question why the square of the angular momentum in quantum mechanics is given by $l(l+1)$ instead of just $l^2$. Make use of these features:

(a) $\langle L^2 \rangle = \langle L_x^2 + L_y^2 + L_z^2 \rangle$ where $\langle \rangle$ refer to average values.

(b) $\langle L_x^2 \rangle = \langle L_y^2 \rangle = \langle L_z^2 \rangle$

(c) $L_z$ (in units of $\hbar$) has $2l+1$ allowed values, namely $m = -l, -l+1, \ldots, l-1, l$.

Express $\langle L^2 \rangle$ in terms of $\sum_{m=-l}^{l} m^2$.

Prove by mathematical induction:

$$\sum_{n=1}^{N} n^2 = \frac{1}{6} N(N+1)(2N+1)$$

Use this result to obtain $\langle L^2 \rangle = l(l+1)$. 