Electricity and Magnetism, ch 24
Physics 208  A.Finkel'stein
Capacitors and capacitance

Capacitor is a sort of storage

Each of the plates of a capacitor is metallic; each of the plates is equipotential

**Capacitance** is measured in Farads

\[
C = \frac{Q}{V_{ab}} \quad \text{(definition of capacitance)}
\]

1 F = 1 farad = 1 C/V = 1 coulomb/volt
Parallel-plate capacitor

Consider two large metal plates which are parallel to each other and separated by a distance $d$ small compared with their width.

The field between plates is

$$E_y = - \frac{\sigma}{\varepsilon_0} = - \frac{Q}{\varepsilon_0 A}$$

$$- [V(top) - V(bottom)] = \int_0^d E_y dy =$$

$$\int_0^d \frac{(-\sigma)}{\varepsilon_0} dy = -\frac{\sigma}{\varepsilon_0} d = \frac{\sigma A d}{\varepsilon_0 A} = - \frac{Qd}{\varepsilon_0 A}$$

$$C = \frac{Q}{V_{ab}} = \varepsilon_0 \frac{A}{d}$$
Capacitance is measured in farads (in honor of Faraday)

the capacitance is the ratio \( \frac{Q}{\Delta V} \)

If charge (Q) is in coulombs, V (voltage) is in volts, the capacitance (C) is in farads

\[ 1F \equiv 1 \text{farad} = 1 \text{C/V} = 1 \text{coulomb/volt} \]

\[ C = \frac{Q}{\Delta V} \sim \varepsilon_0 L \]

A/d is in meters, therefore, farad can be related to other units. How to reconcile farad with other units?
Units:
electric field is measured in \( \text{N/C} \)
electric potential is thus measured in \( (\text{N/C})\text{m}=\text{V} \)
electric field is, therefore, measured in: \( \text{V/m}=\text{N/C} \)

1 volt = 1 \( (\text{N/C})\text{m}=1 \text{ J/C}=1 \text{ joule/coulomb} \)

1 V = 1 \( (\text{N/C})\text{m} \)

How to reconcile farad with other units?

1 F = 1 C/V = 1 C^2/(Nm) = 1 C^2/J

\[
1/(4\pi\varepsilon_0) = 9 \times 10^9 \text{N m}^2/\text{C}^2 = 9 \times 10^9 \text{m/F}
\]

\[
\varepsilon_0 = 8.85 \times 10^{-12} \text{ F/m}
\]
If charge (Q) is in coulombs, V (voltage) is in volts, the capacitance (C) is in farads

\[ C = \frac{Q}{\Delta V} = \frac{A\varepsilon_0}{d} \]

\[ \varepsilon_0 = 8.85 \times 10^{-12} F/m \]

Similarly to 1 Coulomb, 1 Farad is absolutely unrealistic (too large) quantity.

Capacitance in real life is measured from picofarads to microfarads:

\[ 1 pF = 10^{-12} F \quad 1 \mu F = 10^{-6} F \]

**Example 24.1 Size of a 1-F capacitor**

\[ d = 1 mm = 10^{-3} m \]
Capacitors and capacitance

\[ C = \frac{Q}{\Delta V} = \frac{A \varepsilon_0}{d} \sim \varepsilon_0 L \]
Spherical capacitor

\[ C \sim \varepsilon_0 L \]

\[
C = \frac{Q}{V_{ab}}
\]

\[
V_{ab} = \int_{R_a}^{R_b} dr E(r) = \int_{R_a}^{R_b} dr \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0 R_a} - \frac{Q}{4\pi\varepsilon_0 R_b}
\]

\[
C = 4\pi\varepsilon_0 \frac{R_a R_b}{R_b - R_a}
\]

(a) \( R_b \rightarrow \infty \quad C = 4\pi\varepsilon_0 R_a \) capacitance of a sphere

(b) \( R_a \approx R_b >> R_b - R_a = d \quad C = \varepsilon_0 \frac{4\pi R^2}{R_b - R_a} = \varepsilon_0 \frac{A}{d} \) parallel-plate capacitor
Capacitance of a human body

\[ C \sim 4\pi\varepsilon_0 L \quad \varepsilon_0 = 8.85 \times 10^{-12} \, F/m \]

\[ C \sim 10^2 \, pF \quad pF = 10^{-12} \, F \quad \mu F = 10^{-6} \, F \]

Farad is a too big unit. Capacitance in real life is measured from picofarads, \( 10^{-12}F \), up to microfarads, \( 10^{-6}F \).

Capacitance of a human body is comparable with those used in radio tuning.

Capacitance used in cameras as a storage of energy for a flash light is of a few hundred microfarads.
Commercial capacitors

24.4 A commercial capacitor is labeled with the value of its capacitance. For these capacitors, $C = 2200 \mu F$, $1000 \mu F$, and $470 \mu F$.

24.13 A common type of capacitor uses dielectric sheets to separate the conductors.
Example 24.4

A cylindrical capacitor

24.6 A long cylindrical capacitor. The linear charge density $\lambda$ is assumed to be positive in this figure. The magnitude of charge in a length $L$ of either cylinder is $\lambda L$. 

Coaxial cable
\[ V(\infty) = 0 \]

**Example 23.10** An infinite line charge or charged conducting cylinder

23.20 Electric field outside (a) a long positively charged wire and (b) a long, positively charged cylinder.

(a) \[ E_r \]

(b) \[ E_r \]

\[ V(r_0) = 0 \]

\[ V(\vec{r}) = \int_{\vec{r}} \vec{E} d\vec{r} = \int_{\vec{r}} \frac{\lambda}{2\pi\epsilon_0} \frac{dy}{y} = \frac{\lambda}{2\pi\epsilon_0} \ln y \bigg|_{r}^{\infty} \]

Only potential difference matters, measure potential difference with respect to an arbitrary point \( r_0 \)
A long cylindrical capacitor. The linear charge density $\lambda$ is assumed to be positive in this figure. The magnitude of charge in a length $L$ of either cylinder is $\lambda L$.

$$V(\vec{r}) = \int_{r_a}^{r_b} \vec{E} \, dr = \int_{r_a}^{r_b} \frac{\lambda}{2\pi\epsilon_0} \frac{dy}{y} = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{y}{r_a} \bigg|_{y=r_b}$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_b}{r_a}$$

The capacitance per unit length is

$$\frac{C}{L} = \frac{2\pi\epsilon_0}{\ln \left( \frac{r_b}{r_a} \right)}$$

A typical cable for TV antennas and VCR connections has a capacitance per unit length of 69 pF/m.
Why it is obvious: capacitors in series and in parallel

\[ C = \frac{Q}{\Delta V} = \varepsilon_0 \frac{A}{d} \]

\[ C = \frac{Q}{\Delta V} = \varepsilon_0 \frac{A}{d} \quad \Rightarrow \quad \varepsilon_0 \frac{A_1 + A_2 + \ldots}{d} \]

Capacitors in parallel: \( C_{tot} = C_1 + C_2 + C_3 + \ldots \)
Why it is obvious: capacitors in series and in parallel

\[
C = \frac{Q}{\Delta V} = \varepsilon_0 \frac{A}{d}
\]

\[
C = \frac{Q}{\Delta V} = \varepsilon_0 \frac{A}{d} \quad \Rightarrow \quad \varepsilon_0 \frac{A}{d_1 + d_2 + \ldots}
\]

Capacitors in series:

\[
\frac{1}{C_{tot}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots
\]
Capacitors in series

\[ \frac{1}{C_{\text{tot}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots \]

\[ C = \frac{Q}{\Delta V} \quad \Delta V = \frac{Q}{C} \]

\[ V_{ac} = V_1 = \frac{Q}{C_1} \quad V_{cb} = V_2 = \frac{Q}{C_2} \]

\[ V_{ab} = V = V_1 + V_2 = Q \left( \frac{1}{C_1} + \frac{1}{C_2} \right) \]

\[ \frac{V}{Q} = \frac{1}{C_1} + \frac{1}{C_2} \]

(b) The equivalent single capacitor

Charge is the same as for the individual capacitors.

Equivalent capacitance is less than the individual capacitances:

\[ C_{eq} = \frac{Q}{V} \]

\[ \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \]
Capacitors in parallel \[ C_{tot} = C_1 + C_2 + C_3 + \ldots \]

\[ C = \frac{Q}{\Delta V} \]

\[ Q = Q_1 + Q_2 = (C_1 + C_2)V \]

\[ \frac{Q}{V} = C_1 + C_2 \]
Network of Capacitors

(a) Replace these series capacitors by an equivalent capacitor...

(b) ...replace these parallel capacitors by an equivalent capacitor

(c) ...replace these series capacitors by an equivalent capacitor.
Capacitor as an energy storage

\[ V = \frac{Q}{C} \quad F = kX \]

\[ Q = CV \quad X = \frac{F}{k} \]

\[ \frac{1}{C} \Leftrightarrow k \]

\[ dW = v dq = \frac{q}{C} dq \quad dW = f dx = k x dx \]

\[ W = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C} \quad W = \int_0^x k x dx = \frac{1}{2} k X^2 \]

\[ W = \text{potential energy} = U = \frac{Q^2}{2C} = \frac{CV^2}{2} \]
Energy density of the electric field

\[ U = \frac{Q^2}{2C} = \frac{CV^2}{2} \]

\[ u = \frac{U}{\text{volume}} = \text{energy density} = \frac{CV^2}{2Ad} \]

\[ u = \frac{\varepsilon_0 A}{2d} \frac{V^2}{Ad} = \frac{\varepsilon_0 V^2}{2d^2} = \frac{1}{2} \varepsilon_0 E^2 \]

Energy of the capacitor is, in fact, stored between the plates. The “vacuum” is filled with the electric field; electric field acts like elastic media.
Energy density of the electric field; spherical capacitor

Example 24.9

\[ W = \text{potential energy} = U = \frac{Q^2}{2C} \]

\[ u(r) = \frac{1}{2} \varepsilon_0 E^2(r) = \frac{1}{2} \varepsilon_0 \left( \frac{Q}{4\pi \varepsilon_0 r^2} \right)^2 \]

\[ U = \int u \, dV = \int_{r_a}^{r_b} \left( \frac{Q^2}{32\pi^2 \varepsilon_0 r^4} \right) 4\pi r^2 \, dr \]

\[ = \frac{Q^2}{8\pi \varepsilon_0} \left[ \frac{1}{r_b} - \frac{1}{r_a} \right] = \frac{Q^2}{8\pi \varepsilon_0} \left( -\frac{1}{r_b} + \frac{1}{r_a} \right) \]

\[ = \frac{Q^2}{8\pi \varepsilon_0} \frac{r_b - r_a}{r_a r_b} \]
24.12 When the switch $S$ is closed, the charged capacitor $C_1$ is connected to an uncharged capacitor $C_2$. The center part of the switch is an insulating handle; charge can flow only between the two upper terminals and between the two lower terminals.

\[ V_0 C_1 = Q_0 = Q_1 + Q_2 \Rightarrow 960 \, \mu C \]

\[ V_1 = \frac{Q_1}{C_1} = V_2 = \frac{Q_2}{C_2} = 80 \text{volt} \]

\[ Q_1 = \frac{2}{3} Q_0 = 640 \, \mu C \quad Q_2 = \frac{1}{3} Q_0 = 320 \, \mu C \]

\[ U_{\text{initial}} = \frac{1}{2} Q_0 V_0 = \frac{1}{2} (960 \times 10^{-6} \, \text{C}) (120 \, \text{V}) = 0.058 \, \text{J} \]

\[ U_{\text{final}} = \frac{1}{2} Q_1 V + \frac{1}{2} Q_2 V = \frac{1}{2} Q_0 V \\
= \frac{1}{2} (960 \times 10^{-6} \, \text{C})(80 \, \text{V}) = 0.038 \, \text{J} \]
Moore’s Law (1965): every 2 years the number of transistors on a chip is doubled

2 billion transistors per chip
Why capacitance used in cameras is so large?

Capacitance used in radio tuning is about hundred picofarads.

Capacitance used in cameras as a storage of energy for a flash light is of a few hundred microfarads.

\[ U = \frac{Q^2}{2C} = C\frac{V^2}{2} \]

Why photo-cameras need a large capacitance?
Because a sources of energy is a battery with a given voltage \( V \).
Most capacitors have a non-conducting material, or dielectric, between their conducting plates. When we insert an uncharged sheet of dielectric between the plates, the potential difference decreases to a smaller value $V$.

$$V_0 = \frac{Q}{C_0} \quad \text{without dielectric}$$

$$V = \frac{Q}{C} \quad \text{with dielectric}$$

$$C > C_0$$

When the space between plates is completely filled by the dielectric, the ratio $K = \frac{C}{C_0}$ is called dielectric constant.
When the space between plates is completely filled by the dielectric, the ratio

\[ K = \frac{C}{C_0} \]

is called dielectric constant. \( K > 1 \)

**permittivity** \( \varepsilon = \varepsilon_0 K \)

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**Table 24.1** Values of Dielectric Constant \( K \) at 20°C

<table>
<thead>
<tr>
<th>Material</th>
<th>( K )</th>
<th>Material</th>
<th>( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vacuum</td>
<td>1</td>
<td>Polyvinyl chloride</td>
<td>3.18</td>
</tr>
<tr>
<td>Air (1 atm)</td>
<td>1.00059</td>
<td>Plexiglas</td>
<td>3.40</td>
</tr>
<tr>
<td>Air (100 atm)</td>
<td>1.0548</td>
<td>Glass</td>
<td>5–10</td>
</tr>
<tr>
<td>Teflon</td>
<td>2.1</td>
<td>Neoprene</td>
<td>6.70</td>
</tr>
<tr>
<td>Polyethylene</td>
<td>2.25</td>
<td>Germanium</td>
<td>16</td>
</tr>
<tr>
<td>Benzene</td>
<td>2.28</td>
<td>Glycerin</td>
<td>42.5</td>
</tr>
<tr>
<td>Mica</td>
<td>3–6</td>
<td>Water</td>
<td>80.4</td>
</tr>
<tr>
<td>Mylar</td>
<td>3.1</td>
<td>Strontium titanate</td>
<td>310</td>
</tr>
</tbody>
</table>
Dielectric constant $K$

$K$ is called dielectric constant, $K > 1$

permittivity $\varepsilon = \varepsilon_0 K$

$E = \frac{\sigma}{\varepsilon}$

to avoid confusion, don’t use $V_0$
Dielectric constant $K$

$$K = \frac{C}{C_0}$$ is called dielectric constant. \( K > 1 \)

permittivity \( \varepsilon = \varepsilon_0 K \)

$$C = \frac{Q}{V} = \varepsilon_0 K \frac{A}{d} = \varepsilon \frac{A}{d}$$

$$E = \frac{\sigma}{\varepsilon} \quad u = \frac{1}{2} \varepsilon E^2$$

Part of the energy is stored by the polarizing molecules. Working with the same battery (fixed voltage source) one may accumulate larger charge. As a result, larger energy accumulation. Charging of the capacitor takes time.
Additional Materials
Transistors, Moore’s Law
Moore’s Law (1965): every 2 years the number of transistors on a chip is doubled

2 billion transistors per chip
MOSFET: the workhorse of Integrated Circuits

Si-MOSFET (metal-oxide-surface field-effect transistor)

gate voltage opens or closes the field effect transistor, which is recharging like any other capacitor
Example 24.7

Voltage, charge and energy

24.12 When the switch $S$ is closed, the charged capacitor $C_1$ is connected to an uncharged capacitor $C_2$. The center part of the switch is an insulating handle; charge can flow only between the two upper terminals and between the two lower terminals.

\[
\begin{align*}
V_0 C_1 &= Q_0 = Q_1 + Q_2 \Rightarrow 960 \mu C \\
V_1 &= \frac{Q_1}{C_1} = V_2 = \frac{Q_2}{C_2} = 80 \text{ volt} \\
Q_1 &= \frac{2}{3} Q_0 = 640 \mu C \\
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\end{align*}
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U_{\text{final}} &= \frac{1}{2} Q_1 V + \frac{1}{2} Q_2 V = \frac{1}{2} Q_0 V \\
&= \frac{1}{2} (960 \times 10^{-6} \text{ C})(80 \text{ V}) = 0.038 \text{ J}
\end{align*}
Intel switches to hafnium-based transistors for new 45-nm Penryn processors.