Rules of the exam:

1. You have the full class period to complete the exam.
2. Formulae are provided on the last page. You may NOT use any other formula sheet.
3. When calculating numerical values, be sure to keep track of units.
4. You may use this exam or come up front for scratch paper.
5. Be sure to put a box around your final answers and clearly indicate your work to your grader.
6. Clearly erase any unwanted marks. No credit will be given if we cant figure out which answer you are choosing, or which answer you want us to consider.
7. Partial credit can be given only if your work is clearly explained and labeled.
8. All work must be shown to get credit for the answer marked. If the answer marked does not obviously follow from the shown work, even if the answer is correct, you will not get credit for the answer.

Sign below to indicate your understanding of the above rules.

Name : __________________________

Student ID : ............................

Signature : .............................
<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slab Polisher</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Breaking Symmetry</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Playing tricks</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Equilibrium</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td><strong>Total:</strong></td>
<td><strong>120</strong></td>
<td></td>
</tr>
</tbody>
</table>
Part 1: Slab Polisher  
A slab of wood has mass $m$ and is atop a flat surface ready to be polished by a big polishing wheel. The wheel is placed above the slab, spinning rapidly, and slowly pressing down against the slab. Even when the wheel is pressed against the slab it rotates while slipping, that is the surface of the wheel and the slab are at movement with respect to each other. The coefficients of friction are indicated in the figure.

(a) (6 points) The wheel is pressing with a force $F$ against the slab, but the slab still does not move with respect to the table. Draw the free body diagram of the slab, indicating all forces relevant to this problem.

(b) (6 points) Express the friction forces in terms of the forces you used in the free-body diagram and the coefficients of frictions.

\[ f_k = \mu_k^w F \]
\[ f_s \leq \mu_s^w N \]
(c) (9 points) Find the minimum magnitude of the force $F$ that the wheel needs to press against the slab so that the slab will start accelerating in the horizontal direction.

\[
\begin{align*}
\hat{y} : & \quad N - F - mg = 0 \Rightarrow N = mg + F \\
\hat{x} : & \quad f_k - f_s = ma_x, \text{ if } a_x > 0 \Rightarrow f_k > f_s \\
& \quad f_k = \mu^w_k F > \mu^i_s N \\
& \quad = \mu^w_k F > \mu^i_s (mg + F) \Rightarrow F(\mu^w_k - \mu^i_s) = \mu^i_s mg \Rightarrow \\
& \quad F > mg \frac{\mu^i_s}{\mu^w_k - \mu^i_s}
\end{align*}
\]

(d) (9 points) After the slab has been put in motion, find the magnitude of the force $F$ that the wheel needs to press against the slab so it will accelerate with acceleration $a_x$.

\[
\begin{align*}
\hat{y} : & \quad N - F - mg = 0 \Rightarrow N = mg + F \\
\hat{x} : & \quad f^w_k - f^i_k = ma_x = \mu^w_k F - \mu^i_k N = \mu^w_k F - \mu^i_k (mg + F) \Rightarrow \\
& \quad ma_x = F(\mu^w_k - \mu^i_k) - \mu^i_k mg \Rightarrow \\
& \quad F = \frac{m(g \mu^i_k + a_x)}{\mu^w_k - \mu^i_k}
\end{align*}
\]
Part 2: Breaking Symmetry

In particle physics the Higgs boson is created through a symmetry-breaking potential of the type \( u(x, y) = p_0(x^2 + y^2 - p_1^2)^2 \) where \( p_0 \) and \( p_1 \) are known positive numbers. Imagine an object moves in the \((x,y)\) plane subjected to that potential. No other forces are present in this problem, and ignore what you know about quantum mechanics.

(a) (10 points) If the object has a total mechanical energy of \( E \), and its kinetic energy when it is at position \((x=0,y=0)\) is \( K_i \). Find the kinetic energy of the object when it is at position \((x = \frac{p_1}{\sqrt{2}}, y = \frac{p_1}{\sqrt{2}})\). Is it greater or smaller than \( K_i \)?

\[
E = K_i + U(0, 0) = K_i + p_0p_1^4
\]
\[
E_f = K_f + U\left(\frac{p_1}{\sqrt{2}}, \frac{p_1}{\sqrt{2}}\right) = K_f
\]

Since \( E_i = E_f = K_i + p_0p_1^4 = K_f \) we get \( K_i + p_0p_1^4 = K_f \)

Obviously \( K_f \) is larger than \( K_i \).

(b) (10 points) Find the force in the horizontal and vertical direction as a function of the \((x,y)\) position of the object, and then all the positions in the plane \((x,y)\) that are equilibrium points. (Hint: you may use radial coordinates)

\[
\vec{F} = -\nabla U = -\left(\frac{dU}{dx}, \frac{dU}{dy}\right) \Rightarrow \begin{cases} F_x = -p_04x(x^2 + y^2 - p_1^2) \\ F_y = -p_04y(x^2 + y^2 - p_1^2) \end{cases}
\]

- Equilibrium points: when \( \vec{F}(x, y) = 0 \), so we get

\[
\begin{cases} F_x = -p_04x(x^2 + y^2 - p_1^2) = 0 \\ F_y = -p_02y(x^2 + y^2 - p_1^2) = 0 \end{cases}
\]

\((x, y) = (0, 0)\)

\[x^2 + y^2 = p_1^2 \Rightarrow \text{or } r = p_1\]
(c) (10 points) If the object has a total mechanical energy of $E$, find the maximum radial distance from the origin that the object can ever reach.

In the allowed range of positions $(x,y)$ the kinetic energy positive

$$K(x, y) \geq 0 \Rightarrow E = K(x, y) + U(x, y) \geq U(x, y)$$

$$E \geq U(x, y) = p_0(x^2 + y^2 - p_1^2) = p_0(r^2 - p_1^2)$$

$$\Rightarrow \sqrt{\frac{E}{p_0} + p_1^2} \geq x^2 + y^2 = r^2. \Rightarrow$$

$$r \leq \sqrt{\frac{E}{p_0} + p_1^2}$$
Part 3: Playing tricks (30 points)
A carton of milk with mass $m_m$ is laying atop a tray of mass $m_t$ and length $d$ as shown in the picture below. A force $F$ is applied to the tray. There is kinetic friction $\mu_k$ between the milk and the tray, and no friction between the tray and that table. No static friction should be considered in this problem.

(a) (3 points) If there were no friction between the tray and the milk, how far from the original horizontal position will the milk land?

If there was no friction the milk will land in exactly the same position because there would be no horizontal force acting on the milk.

(b) (5 points) Find the work done by that friction force until the tray is completely removed from behind the milk?

From the free-body diagram we get:
\[ \dot{y} : \quad N_1 - m_m g = 0 \Rightarrow N_1 = m_m g \]
\[ \dot{x} : \quad f_k = m_m a_m \]
The work of the friction forces are:
\[ W_{f_k \text{ on milk}} = \vec{f}_k \cdot (\vec{x}_f - \vec{x}_i) = \mu_k N_1 (L - d) = \mu_k m_m g (L - d) \]
\[ W_{f_k \text{ on tray}} = \vec{f}_k \cdot (\vec{x}_f - \vec{x}_i) = -\mu_k N_1 L = -\mu_k m_m g L \]
\[ W_{f_k \text{ total}} = W_{f_k \text{ on milk}} + W_{f_k \text{ on tray}} = -\mu_k m_m g d \]
(c) (10 points) Find the acceleration of the milk and that of the tray.

For the Milk:
\[ \dot{y} : \quad N_1 - m_m g = 0 \Rightarrow N_1 = m_m g \]
\[ \hat{x} : \quad f_k = m_m a_m \Rightarrow a_m = \mu_m g \]

For the Tray:
\[ \dot{y} : \quad N_2 - N_1 - m_t g = 0 \Rightarrow N_2 = (m_m + m_t) g \]
\[ \hat{x} : \quad F - f_k = F - \mu_k m_m g = m_t a_t \Rightarrow a_t = \frac{F - \mu_k m_m g}{m_t} \]

(d) (6 points) If takes a time \( \tau \) to slide the tray all way underneath the milk carton, find the velocity of the tray and milk at that point.

\[ v_m(\tau) = v_{0,m} + a_m t = \mu_k g \tau \]
\[ v_t(\tau) = v_{0,m} + a_t t = \frac{F - \mu_k m_m g}{m_t} \tau \]

(e) (6 points) Using the work energy theorem find the distance \( L \) that the tray had to move before the milk is out of the tray.

\[ K_{i,milk} + K_{i,tray} + W_F + W_{f_k + total} = K_{f,milk} + K_{f,tray} \]
\[ F L - \mu_k m_m g d = K_{f,milk} + K_{f,tray} \Rightarrow L = \frac{K_{f,milk} + K_{f,tray} + \mu_k m_m g d}{F} \]
Part 4: Equilibrium  
(30 points)
The system depicted below is in equilibrium. The Forces $F_1$ and $F_2$ and the angle $\beta$ are known, but the tensions $T_1$, $T_2$, $T_3$, and the angle $\alpha$ are not.

(a) (15 points) Write all the equations that you would need to completely solve this problem. Don’t solve the system yet.

For P1:
\[
\begin{align*}
\hat{x} & : -F_1 + T_2 \sin \alpha = 0 \\
\hat{y} & : T_1 - T_2 \cos \alpha = 0
\end{align*}
\]

For P2:
\[
\begin{align*}
\hat{x} & : T_3 \sin \beta - T_2 \sin \alpha = 0 \\
\hat{y} & : T_2 \cos \alpha + T_3 \cos \beta - F_2 = 0
\end{align*}
\]

(b) (15 points) Solve the system of equations from the above part to obtain an expression for the angle $\alpha$ depending only on $F_1$, $F_2$, and $\beta$.

From the first equation above we get $T_2 = \frac{F_1}{\sin \alpha}$, so replacing we get
\[
\begin{align*}
\hat{y} : T_1 = \frac{F_1}{\sin \alpha} \cos \alpha \\
\hat{x} : T_3 \sin \beta = \frac{F_1}{\sin \alpha} \sin \alpha = F_1 \Rightarrow T_3 = \frac{F_1}{\sin \beta} \\
\hat{y} : \frac{F_1}{\sin \alpha} \cos \alpha + T_3 \cos \beta = F_2
\end{align*}
\]

And replacing now $T_3 = \frac{F_1}{\sin \beta}$ in the last equation we get
\[
\frac{F_1}{\tan \alpha} = F_2 - \frac{F_1}{\tan \beta} \Rightarrow \tan \alpha = \frac{F_1}{F_2 - \frac{F_1}{\tan \beta}}
\]