Problem 1: (5 points)

Write Maxwell's equations in the integral form.

\[ \oint E \cdot d\mathbf{s} = \frac{Q}{\varepsilon_0} \]
\[ \oint B \cdot d\mathbf{s} = 0 \]
\[ \oint E \cdot d\mathbf{r} = -\frac{d}{dt} \oint B \cdot d\mathbf{s} \]
\[ \oint B \cdot d\mathbf{r} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d}{dt} \oint E \cdot d\mathbf{s} \]
Problem 2: (15 points)

Electric charge is distributed uniformly along each side of a square with length $a$. Two adjacent sides have positive charge with total charge $+Q$ on each.

a) If the other two sides have negative charge with total charge $-Q$ on each (see the figure below), what is the net electric field at the center of the square?

\[ dE_y = -\frac{1}{4\pi\varepsilon_0} \frac{dQ}{r^2} \sin \theta \]

\[ \sin \theta = \frac{a}{2r} \quad ; \quad r = \sqrt{a^2 + \frac{a^2}{4}} \]

\[ E_y = -\frac{a}{4\pi\varepsilon_0} \int_{0}^{a} \frac{Q}{\alpha} \frac{dx}{\alpha \left(a^2 + \frac{a^2}{4}\right)^{3/2}} \]

\[ = -\frac{Q}{4\pi\varepsilon_0} \frac{a}{4} \left(\frac{a^2}{4} + \frac{a^2}{4}\right)^{1/2} = -\frac{Q}{4\pi\varepsilon_0} \frac{a}{4} \left(\frac{2a^2}{4}\right)^{1/2} = -\frac{Q}{4\pi\varepsilon_0} \frac{a}{4} \frac{a}{\sqrt{2}} = -\frac{Q}{\sqrt{2}\varepsilon_0 \sqrt{a^2}} \]

\[ E = -\frac{\sqrt{2}}{\varepsilon_0 \sqrt{1}} \frac{Q}{a^2} \left(\hat{i}_x + \hat{i}_y\right) \]

b) Find the electric field in the center of the square if all four sides have positive charge $+Q$. 

\[ E = 0 \]
Problem 3: (20 points)

a) A non-uniform, but spherically symmetric, distribution of charge has a charge density \( \rho(r) \) given as follows:
\[
\rho(r) = \rho_0 \left(1 - \frac{r}{R}\right) \quad \text{for } r \leq R \\
\rho(r) = 0 \quad \text{for } r \geq R
\]

Find the electric field everywhere.

\( r < R \)
\[
Q_{\text{encl}} = \int_0^r \rho_0 (1 - \frac{r}{R}) 4\pi r^2 dr = \frac{Q_0}{2} \left(\frac{R^3}{3} - \frac{r^3}{3}\right)
\]
\[E = \frac{Q_0}{\pi \varepsilon_0 R^2} \left(\frac{r^3}{3} - \frac{R^3}{3}\right) \quad \text{radially out}
\]

\( r > R \)
\[
Q_{\text{encl}} = \int_0^R \rho_0 (1 - \frac{r}{R}) 4\pi r^2 dr = \frac{Q_0}{2} \left(\frac{R^3}{3} - \frac{r^3}{3}\right)
\]
\[E = \frac{Q_0}{\pi \varepsilon_0 R^2} \left(\frac{r^3}{3} - \frac{R^3}{3}\right) \quad \text{radially out}
\]

b) A solid conducting sphere with radius R, that carries positive charge \( Q \), is concentric with a very thin conducting shell of radius \( 2R \) that carries charge \( -2Q \). Find the electric field (magnitude and direction) in each of the regions \( 0 < r < R \), \( R < r < 2R \), and \( r > 2R \). Draw schematically the electric field lines.

\[E = \begin{cases} 0 & r < R \\
\frac{Q}{\pi \varepsilon_0 r^2} & \text{radially out} \quad R < r < 2R \\
-\frac{Q}{\pi \varepsilon_0 r^2} & \text{radially in} \quad r > 2R
\end{cases}
\]

c) For part b) find a difference in electric potential between points \( r = 0 \) and \( r = \infty \)
\[
V(0) - V(\infty) = -\int_0^\infty E \cdot dr = \int_0^R \frac{Q}{\pi \varepsilon_0 r^2} dr - \int_R^\infty \frac{Q}{\pi \varepsilon_0 r^2} dr = \frac{Q}{\pi \varepsilon_0 R} - \frac{Q}{\pi \varepsilon_0 2R}
\]
\[= 0
\]
Problem 4: (20 points)

Two horizontal conducting rails form an angle $\theta$ where their ends are joined. A conducting rod slides on the rails without friction with constant velocity $v$ as shown in the figure. At $t=0$ it starts from $x=d$ from rest. There is a resistor $R$ in one of the rails. There is a uniform magnetic field in the direction shown.

(a) Find the current as a function of time. Ignore self-inductance.

\[
\Phi = \int B \cdot ds = B \frac{1}{2} x^2 \tan \theta
\]

\[
x = d + vt
\]

\[
\Phi = B \frac{1}{2} (d + vt)^2 \tan \theta
\]

\[
\frac{d\Phi}{dt} = B (d + vt) vt \tan \theta
\]

\[
\Phi \vec{E} \cdot d\vec{r} = -\frac{d\Phi}{dt}
\]

\[
-iR = -B (d + vt) vt \tan \theta
\]

\[
i = \frac{B (d + vt) vt \tan \theta}{R} \text{ clockwise}
\]

(b) Find the force that needs to be applied to move the rod with constant velocity.

\[
\vec{F} = i \vec{E} \times \vec{B} = iE \vec{B} = (d + vt) \tan \theta iB \frac{\vec{e}_x}{x}
\]

\[
e = x \tan \theta = (d + vt) \tan \theta
\]
Problem 5: (15 points)

Show that the displacement current between the plates of a parallel plate capacitor is given by $\frac{C}{dt}$, where $C$ is capacitance and $V$ is the potential difference between the plates.

\[
\begin{align*}
\frac{d}{dt} = \varepsilon_0 \frac{d}{dt} \int E \cdot dS &= \varepsilon_0 \frac{d}{dt} E \cdot A = \\
&= \varepsilon_0 A \frac{dE}{dt} = \varepsilon_0 A \frac{d}{dt} \frac{V}{d} = \\
&= \varepsilon_0 A \frac{dV}{dt} = C \frac{dV}{dt}
\end{align*}
\]
Problem 6: (15 points)

In the circuit below the switch is closed at \( t = 0 \).

\[ E \cdot d\vec{r} = -\frac{d\Phi}{dt} \]

\( \Phi = \pm BLi; \quad \Phi = -BLi; \quad \frac{d\Phi}{dt} = -BL \frac{di}{dt} \]

\[ V - iR = +BL \frac{di}{dt} \]

\[ L \frac{di}{dt} + Ri = V \]

\[ i(t) = \frac{V}{R} + \alpha \frac{R}{L} t \]

\[ L \frac{di}{dt} + Ri = 0 \]

\[ i = \alpha e^{\beta t} - \alpha \frac{R}{L} e^{\beta t} + R \alpha e^{\beta t} = 0 \]

\[ \beta = \frac{R}{L} \]

Find the current as a function of time and plot it schematically.

\[ i(0) = 0 = \frac{V}{R} + \alpha = 0 \]

\[ \alpha = -\frac{V}{R} \]

\[ i(t) = \frac{V}{R} (1 - e^{\beta t}) \]
Problem 7: (15 points)

The circuit below was put together a long time ago so that the steady state has been reached.

\[
\begin{align*}
- V_1 + iR_1 + V_C + iR_2 &= 0 \\
- V_1 + iR_1 + \frac{Q}{C} &= 0 \\
\end{align*}
\]

\[
\left\{ \begin{array}{l}
- V_1 + i R_1 + V_C + i R_2 = 0 \\
- V_1 + i R_1 + \frac{Q}{C} = 0 \\
\end{array} \right. \Rightarrow i = \frac{V_1 - V_2}{R_1 + R_2}
\]

\[
\frac{Q}{C} = V_1 - i R_1 = V_1 - \frac{V_1 - V_2}{R_1 + R_2} R_1
\]