1. (25 points) Two charges are fixed at the positions shown.

The distances $a$ and $b$ are known. The charge at the origin is negative, $-q_1$. The charge $q_2$ at $x = a, y = b$ is positive. Find the force that would be exerted on a charge $q_3$ if it were placed at an arbitrary point $x, y$.

**Law**

$$\vec{F} = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r^2} \vec{r}$$

**Application**

$$F_{1x} = -\frac{1}{4\pi \varepsilon_0} \frac{q_1 q_3}{x^2+y^2} \cos \theta$$

$$F_{1y} = -\frac{1}{4\pi \varepsilon_0} \frac{q_1 q_3}{x^2+y^2} \sin \theta$$

$$F_{2x} = -\frac{1}{4\pi \varepsilon_0} \frac{q_2 q_3}{(a-x)^2+(y-b)^2} \cos \varphi$$

$$F_{2y} = -\frac{1}{4\pi \varepsilon_0} \frac{q_2 q_3}{(a-x)^2+(y-b)^2} \sin \varphi$$

$$\vec{F} = \left[ F_{1x} + F_{2x} \right] \hat{i}_x + \left[ F_{1y} + F_{2y} \right] \hat{i}_y$$

**Result**

$$\vec{F} = \left[ -\frac{1}{4\pi \varepsilon_0} \frac{q_1 q_3}{(x^2+y^2)^{3/2}} - \frac{1}{4\pi \varepsilon_0} \frac{q_2 q_3}{(a-x)^2+(y-b)^2}^{3/2} \right] \hat{i}_x$$

$$+ \left[ -\frac{1}{4\pi \varepsilon_0} \frac{q_1 q_3}{(x^2+y^2)^{3/2}} + \frac{1}{4\pi \varepsilon_0} \frac{q_2 q_3}{\sqrt{(a-x)^2+(y-b)^2}} \right] \hat{i}_y$$
2. (25 points) There is a charged semi-circle of radius $R$. From $\theta = 0$ to $\theta = \frac{\pi}{2}$ there is a charge $Q_1$ uniformly distributed. From $\theta = \frac{\pi}{2}$ to $\theta = \pi$ the charge is not uniformly distributed but instead the charge per unit length is a function of $\theta$ given by $\lambda(\theta) = \frac{Q_2}{\pi} \frac{\theta}{\pi}$. Find the electric field at the center of the semi-circle.

\[ \vec{E} = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} \vec{r} \]

**Application**

\[
\begin{align*}
    dQ_1 &= \frac{2Q_1}{\pi R} R d\theta \\
    dE_{x1} &= \frac{1}{4\pi \varepsilon_0} \frac{dQ_1}{R^2} \sin \theta \\
    dE_{y1} &= -\frac{1}{4\pi \varepsilon_0} \frac{dQ_1}{R^2} \cos \theta
\end{align*}
\]

\[
\begin{align*}
    E_{x1} &= \frac{1}{4\pi \varepsilon_0} \int_0^{\frac{\pi}{2}} \frac{2Q_1}{\pi R} \cos(\theta - \frac{\pi}{2}) d\theta \\
    &= -\frac{1}{4\pi \varepsilon_0} \frac{2Q_1}{R^2} \sin \theta \bigg|_0^{\frac{\pi}{2}} \\
    &= \frac{1}{2\pi \varepsilon_0} \frac{Q_1}{R^2} \frac{\pi}{2}
\end{align*}
\]

\[
\begin{align*}
    E_{y1} &= -\frac{1}{4\pi \varepsilon_0} \frac{2Q_1}{\pi R^2} \int_0^{\frac{\pi}{2}} \sin(\theta - \frac{\pi}{2}) d\theta \\
    &= -\frac{Q_1}{2\pi \varepsilon_0} \left( \sin \frac{\pi}{2} - \sin 0 \right) \\
    &= -\frac{Q_1}{2\pi \varepsilon_0 R^2}
\end{align*}
\]

Result \[ [Q_2] = \frac{C}{M} \]

\[ \vec{E} = (E_{x1} + E_{x2}) \hat{i}_x + (E_{y1} + E_{y2}) \hat{i}_y \]
3. (25 points) A positive point charge, $Q_1$, is located at the origin and a negative point charge, $-Q_2$, is located at $x = a, y = b$. Find the difference in the total electric potential function at the two points $x = 0, y = 2b$ and $x = 0, y = -2b$.

\[
V(r) = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r}
\]

**Application**

\[
V_1(x, y) = \frac{1}{4\pi \varepsilon_0} \frac{Q_1}{\sqrt{x^2 + y^2}}
\]

\[
V_2(x, y) = -\frac{1}{4\pi \varepsilon_0} \frac{Q_2}{\sqrt{(a-x)^2 + (y-b)^2}}
\]

\[
V(x, y) = \frac{1}{4\pi \varepsilon_0} \left( \frac{Q_1}{\sqrt{x^2 + y^2}} - \frac{Q_2}{\sqrt{(a-x)^2 + (y-b)^2}} \right)
\]

\[
V(0, -2b) - V(0, 2b) = \frac{1}{4\pi \varepsilon_0} \left( \frac{Q_1}{\sqrt{0 + 4b^2}} - \frac{Q_2}{\sqrt{a^2 + 4b^2}} \right) = \frac{Q_1}{\sqrt{0 + 4b^2}} + \frac{Q_2}{\sqrt{a^2 + 4b^2}}
\]

**Result**

\[
= \frac{1}{4\pi \varepsilon_0} \left( \frac{Q_1}{\sqrt{a^2 + 4b^2}} - \frac{Q_2}{\sqrt{a^2 + 9b^2}} \right) = \frac{Q_1}{4\pi \varepsilon_0} \left( \frac{1}{\sqrt{a^2 + 4b^2}} - \frac{Q_2}{\sqrt{a^2 + 9b^2}} \right)
\]
4. (25 points) A cube with sides of length $a$ is located with one corner at the origin. First find the flux of $\vec{E}$ through the shaded side of the cube if the electric field is given by

$$\vec{E} = \alpha x^2 \vec{i}_x + \beta y^2 \vec{i}_y$$

where $\alpha$ and $\beta$ are known constants. Then find the flux through the shaded side of the cube if the electric field is given by

$$\vec{E} = \alpha y^2 \vec{i}_x + \beta x^2 \vec{i}_y$$

Law

$$\Phi = \int \vec{E} \cdot d\vec{s}$$

Application

$$\vec{s} = a^2 \vec{i}_x, \quad \Phi = da^2 a^2 = da^4$$

$$d\vec{s} = ady \vec{i}_x, \quad E_x = dy^2$$

$$\Phi = \int_0^a \int_0^a \alpha y^2 \cdot ady = da \frac{y^3}{3} \bigg|_0^a = \frac{da^4}{3}$$

Result