Problem 1: (5 points)

Write Maxwell's equations in the integral form.

\[ \oint E \cdot d\mathbf{s} = \frac{Q_{\text{enc}}}{\varepsilon_0} \]
\[ \oint B \cdot d\mathbf{s} = 0 \]
\[ \oint \mathbf{E} \cdot d\mathbf{r} = -\frac{\partial}{\partial t} \oint \mathbf{B} \cdot d\mathbf{s} \]
\[ \oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 (i + \varepsilon_0 \frac{\partial}{\partial t} \oint \mathbf{E} \cdot d\mathbf{s}) \]
Problem 2: (20 points)

A spherical conducting shell, inner radius \( A \) and outer radius \( B \), is charged with charge \( Q_0 \). It is surrounded by a conducting spherical shell of inner radius \( C \) and outer radius \( D \), which is charged with charge \(-2Q_0\).

a) Find the charge per unit area on all surfaces.

\[
\begin{array}{c|c}
 r = A & Z = \frac{Q_0}{4\pi B^2} \\
 r = B & Z = 0 \\
 r = C & Z = -\frac{Q_0}{4\pi C^2} \\
 r = D & Z = -\frac{Q_0}{4\pi D^2}
\end{array}
\]

b) Find the electric field at

i) \( r < A \)

\[ E = 0 \quad (\text{conductor}) \]

ii) \( A < r < B \)

\[ E = 0 \quad (\text{conductor}) \]

iii) \( B < r < C \)

\[ \vec{E} \cdot d\vec{S} = \frac{Q_{\text{encl}}}{\varepsilon_0}; \quad E = \frac{Q_0}{\varepsilon_0}; \quad \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q_0}{r^2} \text{ rad. out} \]

iv) \( C < r < D \)

\[ E = 0 \quad (\text{conductor}) \]

v) \( r > D \)

\[ E = \frac{Q_0}{4\pi\varepsilon_0} \quad \text{rad out} \]

c) Sketch the electric field lines.

d) Find the difference in electric potential between \( r = 0 \) and \( r = \infty \), \( V(\infty) - V(0) \).

\[
V(\infty) - V(0) = -\int_0^\infty \vec{E} \cdot d\vec{r} = -\int_0^A \vec{E} \cdot d\vec{r} - \int_A^B \vec{E} \cdot d\vec{r} - \int_B^C \vec{E} \cdot d\vec{r} - \int_C^D \vec{E} \cdot d\vec{r}
\]

\[
= -\left[ -\frac{Q_0}{4\pi\varepsilon_0} \frac{1}{r} \bigg|_0^A + \frac{Q_0}{4\pi\varepsilon_0} \frac{1}{r} \bigg|_B^\infty \right] - \frac{Q_0}{4\pi\varepsilon_0} \frac{1}{r} \bigg|_C^D
\]

\[
= \frac{Q_0}{4\pi\varepsilon_0} \left[ \frac{1}{A} - \frac{1}{B} + \frac{1}{D} \right]
\]
d) A long, nonconducting, solid cylinder of radius $R$ has a nonuniform volume charge density $\rho = cr^2$.

a) Find the electric field at

1) $r < R$

\[ \Phi E \cdot d\vec{S} = \frac{Q_{ensel}}{\varepsilon_0} \]

\[ E 2\pi r \ell = \frac{1}{\varepsilon_0} \int_S q dV = \frac{1}{\varepsilon_0} \int_0^r 2\pi r^2 2\pi r^2 dr \]

\[ E 2\pi r \ell = \frac{1}{\varepsilon_0} \left( \frac{c r^2 \ell}{4} \right) \]

\[ E = \frac{c r^2 \ell}{4\varepsilon_0} \text{ rad. out} \]

2) $r > R$

\[ E 2\pi r \ell = \frac{1}{\varepsilon_0} \int_0^R 2\pi r^2 \rho r dr \]

\[ E 2\pi r \ell = \frac{c R \ell}{4\varepsilon_0} \]

\[ E = \frac{c R \ell}{4\varepsilon_0} \text{ rad. out} \]
Problem 3: (14 points)

a) Two very long parallel wires are separated by distance $d$ (see the figure below). Current in wire 1 has magnitude $i$ and is out of the page. Current in wire 2 has the same magnitude $i$ and is into the page. In unit-vector notation, what is the net magnetic field at point $P$ at distance $R$ due to the two currents?

\[
\phi \mathbf{B} \cdot d\mathbf{r} = \mu_0 i; \quad \mathbf{B} = \frac{\mu_0 i}{2\pi r}; \quad r = \sqrt{R^2 + \frac{d^2}{4}}.
\]

\[\sum B_y = 0 \text{ from symmetry}\]

\[
B_x = 2\frac{\mu_0 i}{2\pi R} \sin \theta
\]

\[
\sin \theta = \frac{d/2}{r} = \frac{d/2}{\sqrt{R^2 + d^2/4}}
\]

\[
B = B_x \mathbf{i}_x + 0 \mathbf{i}_y
\]

b) A negatively charged particle is injected at point $P$ with a velocity $\mathbf{v} = v_0 \mathbf{i}_x$, where $v_0$ is a constant. What constant electric field (magnitude and direction) would have to be applied for the particle to experience no net force? Ignore gravity.

\[
q \mathbf{E} - q \mathbf{B} = 0
\]

\[
\mathbf{E} = v_0 \mathbf{B}
\]

[c) A positively charged particle is injected at point $P$ with a velocity $\mathbf{v} = v_0 \mathbf{i}_y$, where $v_0$ is a constant. What constant electric field (magnitude and direction) would have to be applied for the particle to experience no net force? Ignore gravity.

\[
q \mathbf{E} - q \mathbf{B} = 0
\]

\[
\mathbf{E} = v_0 \mathbf{B}
\]
Problem 4: (18 points)

At \( t = 0 \) a rectangular loop of wire with length \( W \), width \( H \), and resistance \( R \) is located at distance \( d \) from an infinitely long wire carrying current \( i_0 \). The loop is moved away from the wire at constant speed \( v_0 \).

a) Find the direction of the current in the loop. Explain your answer within this box:

\[ B \text{ decreases, } P_B \text{ decreases, } B_{\text{self}} \text{ is in the same direction as } B_{\text{original}}, \text{ induced is } \text{CW} \]

b) Find the current induced in the loop as a function of time. Ignore self-inductance.

\[
\Phi B \cdot dr = \mu_0 i_0 \cdot \vec{B} = \frac{\mu_0 i_0}{2\pi y} \phi E \cdot dr = -\frac{2}{\phi} \int \vec{B} \cdot ds
\]

\[
\Phi = \int \vec{B} \cdot ds = -\int \frac{\mu_0 i_0}{2\pi y} Wdy = -\frac{\mu_0 i_0}{2\pi} W \left( y_H(y+H) - hy \right)
\]

\[
\frac{d\Phi}{dt} = -\frac{\mu_0 i_0 W}{2\pi} \left( \frac{1}{y+H} \frac{dy}{dt} - \frac{1}{y} \frac{dy}{dt} \right) ; y = d + v_0t
\]

\[
-iR = \frac{\mu_0 i_0 W}{2\pi} \left( \frac{1}{y+H} - \frac{1}{y} \right) v_0
\]

\[
i = \frac{\mu_0 i_0 W}{2\pi R} \left( \frac{1}{y} - \frac{1}{y+H} \right) v_0 \text{ CW}
\]

b) Find the current induced in the loop if the loop is moving along the wire. Ignore self-inductance.

\[ i = 0 \quad \frac{d\Phi_B}{dt} = 0 \]
Problem 5: (18 points)

In the circuit below the fuse has zero resistance as long as the current through it remains less than $i_0$. If the current reaches $i_0$, the fuse "blows" and thereafter has infinite resistance. Switch $S$ is closed at $t=0$.

a) Find the current as a function of time.

$$\Phi = -LI, \quad \frac{d\Phi}{dt} = -L \frac{di}{dt}$$

$$V = L \frac{di}{dt}, \quad \frac{di}{dt} = \frac{V}{L}, \quad i = \frac{V}{L} t + \text{Const}$$

$$i = \frac{V}{L} t$$

b) Find time $t_1$ when the fuse blows.

$$\frac{V}{L} t_1 = i_0 \quad \Rightarrow \quad t_1 = \frac{L i_0}{V}$$

c) Redefine the moment of time when the fuse blows as $t=0$. The current in the circuit at $t=0$ is $i_0$. Find the current in the circuit as a function of time.

$$\Phi = -LI, \quad \frac{d\Phi}{dt} = -L \frac{di}{dt}$$

$$V - iR = L \frac{di}{dt}, \quad L \frac{di}{dt} + Ri = V$$

$$i = i_{ss} + i_H$$

$$i_{ss} = \frac{V}{R}$$

$$i_H(t) = \alpha e^{-\beta t}$$

$$\alpha(-\beta) e^{-\beta t} + \frac{R}{L} \alpha e^{-\beta t} = 0$$

$$\beta = \frac{R}{L}, \quad \alpha = \frac{V}{R}$$

$$i_H(t) = \frac{V}{R} e^{-\beta t}$$

$$i(t) = \frac{V}{R} + i_H = \frac{V}{R} + \alpha e^{-\beta t}$$

$$i(t=0) = -\frac{V}{R} + \alpha = i_0$$
Problem 6: (20 points)

A single loop of wire with an area $A$ is placed in the time varying magnetic field directed into the page. The magnitude of the magnetic field is given by $B = B_0 + \alpha t$, where $B_0$ and $\alpha$ are constants and $t$ is time. The loop has a capacitor, capacitance $C$ that was initially uncharged. It also has resistance $R$.

a) Starting from some famous law, derived the equation that could be solved to find the charges on the capacitor as a function of time if the self-inductance of the loop is $L$. Do not solve it. Please note that without a direction of current and charges on the capacitor indicated on the circuit the problem will not be graded.

\[
\Phi \mathbf{B} \cdot d\mathbf{r} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S}
\]

\[
\dot{Q}_{\text{ext}} + \int \mathbf{B} \cdot d\mathbf{S} = -(B_0 + \alpha t) A
\]

\[
\frac{d}{dt} Q_{\text{self}} = \dot{Q}_{\text{self}} = L \dot{i}
\]

\[
\frac{d}{dt} Q = -[(\alpha A + L \frac{d}{dt})i] = \alpha A - L \frac{di}{dt}
\]

\[
\dot{Q} = \frac{d}{dt} i / C + R \frac{d}{dt} Q = \alpha A - L \frac{d^2 Q}{dt^2}; \quad \frac{L}{R} \frac{d^2 Q}{dt^2} + \frac{R}{C} \frac{d}{dt} Q = \alpha A
\]

b) Ignore the self-inductance. Find the charges on the capacitor as a function of time.

\[
R \frac{dQ}{dt} + \frac{1}{C} Q = \alpha A
\]

\[
\dot{Q}(t) = \dot{Q}_{ss} + \dot{Q}_e
\]

\[
\dot{Q}_{ss} = C \alpha A
\]

\[
\dot{Q}_e = \dot{e} \beta t
\]

\[
R \dot{e} (-\beta) e^{-\beta t} + \frac{1}{C} \dot{e} e^{-\beta t} = 0
\]

\[
\beta = \frac{1}{RC}
\]

\[
\dot{Q}(t) = C \alpha A + \dot{e} e^{-\beta t}
\]

\[
\dot{Q}(t=0) = C \alpha A + \dot{e} = 0
\]

\[
\alpha = -C \alpha A
\]
Problem 7: (7 points)

A capacitor is connected to a battery with \( V = V_m \sin \omega t \). \( V_m \) and \( \omega \) are given constants. The maximum value of the displacement current is \( i_d \). Ignore the resistance and self-inductance of the circuit.

a) What is the maximum value of the current \( i \) in the circuit?

\[ \max i = \max i_d = \frac{V}{d} \]

b) Find the displacement current between the plates, \( i_d \) as a function of time if the distance between the plates is \( d \) and the area of the plates is \( A \).

\[ i_d = \varepsilon_0 \frac{\partial}{\partial t} \int S \vec{E} \cdot d\vec{S} \]

\[ \Phi = \int \vec{E} \cdot d\vec{S} = \frac{V}{d} A = \frac{V_m}{d} \sin \omega t \cdot A \]

\[ \int \vec{E} \cdot d\vec{r} = -\left[ V(r_2) - V(r_1) \right] \]

\[ E \, d = \Delta V, \quad E = \frac{\Delta V}{d} \]

\[ \frac{d\Phi}{dt} = \frac{V_m}{d} A \omega \cos \omega t + \]

\[ i_d = \varepsilon_0 \frac{A}{d} V_m \omega \cos \omega t \]