Problem 1: (5 points)

Write Maxwell’s equations in the integral form.

\[ \oint E \cdot d\overrightarrow{S} = \frac{Q_{\text{enc}}}{\varepsilon_0} \]
\[ \oint B \cdot d\overrightarrow{S} = 0 \]
\[ \oint \varepsilon \cdot d\overrightarrow{r} = -\frac{2}{\varepsilon_0} \int \overrightarrow{B} \cdot d\overrightarrow{S} \]
\[ \oint \overrightarrow{B} \cdot d\overrightarrow{r} = \mu_0 \left( i + \varepsilon_0 \frac{\partial}{\partial t} \oint E \cdot d\overrightarrow{S} \right) \]
Problem 2: (24 points)

a) A conducting spherical shell has inner radius A and thickness T. There is a larger concentric spherical conducting shell with inner radius B and thickness T. The inner shell is given a charge \(-Q\).

   a) Find the charge per unit area on all surfaces.

   \[
   \begin{align*}
   r = A & \quad \beta = 0 \\
   r = A + T & \quad \beta = -\frac{Q}{4\pi (A + T)^2} \\
   r = B & \quad \beta = \frac{Q}{4\pi B^2} \\
   r = B + T & \quad \beta = -\frac{Q}{4\pi (B + T)^2}
   \end{align*}
   \]

   \[\oint \vec{E} \cdot d\vec{S} = \frac{Q_{\text{encl}}}{\varepsilon_0}\]

b) Find the electric field at

i) \( r < A \)

\[\vec{E} = 0 \quad (Q_{\text{encl}} = 0)\]

ii) \( A < r < A + T \)

\[\vec{E} = 0 \quad (\text{conductor})\]

iii) \( A + T < r < B \)

\[\vec{E} = \frac{Q}{4\pi \varepsilon_0} \vec{r} \]

iv) \( B < r < B + T \)

\[\vec{E} = 0 \quad (\text{conductor})\]

v) \( r > B + T \)

\[\vec{E} = \frac{Q}{4\pi \varepsilon_0} \vec{r} \]

c) Sketch the electric field lines.

d) Find the difference in electric potential between \( r = A \) and \( r = B \): \( V(B) - V(A) \).

\[V(B) - V(A) = -\int_A^B \vec{E} \cdot d\vec{r} = +\int_A^B \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} \, dr = \]

\[= -\left. \frac{1}{4\pi \varepsilon_0} \frac{Q}{r} \right|_{A + T}^{B} = -\frac{1}{4\pi \varepsilon_0} Q \left( \frac{1}{B} - \frac{1}{A + T} \right)\]

Result: \( V(B) - V(A) = \)

\[\frac{Q}{4\pi \varepsilon_0} \left( \frac{1}{A + T} - \frac{1}{B} \right)\]
e) A very, very long insulating cylinder of radius \( A \) and length \( L \) has a charge \( Q \) uniformly spread throughout its volume. Consider only points very far from the ends so that the cylinder can be assumed to be infinitely long.

i) Find the electric field as a function of \( r \), the distance from the axis of the cylinder, for all values of \( r \).

\[
\oint E \cdot dS = \frac{Q_{\text{enc}}}{\varepsilon_0}
\]

1) \( r < A \)
\[
E = \frac{1}{2\pi \varepsilon_0} \frac{Q}{\pi A^2 L} \frac{1}{r^2} \text{radially out}
\]

Result: 
- \( r < A \) \( E = \frac{Q}{2\pi \varepsilon_0 A^2 L} \frac{1}{r} \) radially out
- \( r > A \) 

2) \( r > A \)
\[
E = \frac{1}{2\pi \varepsilon_0 L} \frac{Q}{r} \text{radially out}
\]

ii) Sketch the magnitude of electric field as a function of \( r \).
Problem 3: (21 points)

In the circuit below the switch has been in the position A for a long time.

a) Find the current through the resistor $R_2$.

\[
\oint \mathbf{E} \cdot d\mathbf{r} = 0
\]

\[
V - iR = 0
\]

\[
i = \frac{V}{R_2}
\]

b) At $t=0$ the switch is moved to the position B. Starting from some famous law, derive the equation that can be solved to find the charges on the capacitor $C_2$ and current through $R_2$, as a function of time.

\[
\Phi = -Li; \quad \frac{d\Phi}{dt} = -L \frac{di}{dt}; \quad i = -\frac{dQ}{dt}
\]

\[
\frac{Q}{C_2} - iR_2 = L \frac{di}{dt}; \quad \frac{Q}{C_2} + R_2 \frac{dQ}{dt} = -L \frac{d^2Q}{dt^2}
\]

\[
\left[ L \frac{d^2 Q}{dt^2} + R_2 \frac{dQ}{dt} + \frac{1}{C_2} Q = 0 \right]
\]

c) Replace $C_2$ by a perfectly conducting wire. Find the current through $R_2$, as a function of time.

\[
L \frac{di}{dt} + R_2 i = 0
\]

\[
i(t) = d e^{-\beta t}
\]

\[
L \left( -\beta \right) e^{-\beta t} + R_2 d e^{-\beta t} = 0
\]

\[
\beta = \frac{R_2}{L}
\]

\[
i(t) = \frac{d}{R_2} e^{-\frac{R_2}{L} t}
\]

\[
i(t=0) = d = \frac{V}{R_2}
\]

\[
i'(t) = \frac{V}{R_2} e^{-\frac{R_2}{L} t}
\]

d) Plot the current as a function of time schematically.
**Problem 4: (22 points)**

A vertically oriented loop with capacitance $C$, resistance $R$, length $l$ and width $w$, falls from a region where the magnetic field of a given magnitude $B$ is horizontal, uniform, and pointing out of the page as shown in the Figure. At $t = 0$ the capacitor was uncharged. Ignore self-inductance. Note that the positive $y$ direction is down.

![Diagram of a loop with magnetic field](image)

a) Find the charges on the capacitor plates and current in the loop as a function of the velocity $v$ if the loop is falling down but is completely within the region having magnetic field $B$.

\[
\oint \mathbf{E} \cdot d\mathbf{r} = -\frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{s}
\]

\[
\frac{d\Phi}{dt} = 0 \Rightarrow i = 0; Q = 0
\]

b) Once the lower segment leaves the region with magnetic field, find the direction of the current in the loop. Explain your answer within this box:

| area decreases | flux decreases, $B_{self} \circ$  
| to oppose the change  
| $i$ is in counterclockwise direction |

c) Find the equation that can be solved to find the charge on the capacitor as a function of the velocity of the loop $v$ once the lower segment leaves the region with magnetic field.

\[
\oint \mathbf{E} \cdot d\mathbf{r} = -\frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{s}
\]

\[
\Phi = \int \mathbf{B} \cdot d\mathbf{s} = Bwy
\]

\[
\frac{d\Phi}{dt} = Bw \frac{dy}{dt} = -Bwv
\]

\[
\frac{Q}{C} + iR = BWv; \quad i = \frac{dQ}{dt}
\]

\[
\frac{1}{C} Q + R \frac{dQ}{dt} = BWv
\]

\[
R \frac{dC}{dt} + \frac{1}{C} Q = BWv
\]
d) Solve for the charge $Q$ and current in the loop.

\[ Q(t) = Q_{\text{particular}} - Q_{\text{homogeneous}} \]

(1): \[ Q(t) = BwVc \]

(2): \[ Q(t) = \lambda \, e^{\beta t} \frac{R}{(-\beta)} \lambda \, e^{\beta t} + \frac{1}{c} \lambda \, e^{\beta t} = 0 \]

\[ \beta = \frac{1}{RC} \]

\[ Q(t) = BwVc + \lambda \, e^{\frac{t}{RC}} \]

\[ Q(t = 0) = BwVc + \lambda = 0 \]

\[ \lambda = -BwVc \]

\[ Q(t) = BwVc \left( 1 - \frac{t}{RC} \right) \]
Problem 5: (21 points)

a) Coaxial cable: Consider an infinitely long cylindrical conductor carrying a current $i$ spread uniformly over its cross section and a cylindrical conducting shell around it with a current $i$ flowing in the opposite direction. It is uniformly spread over the cross section of the shell.

Find the magnetic field at

i) $r < a$

$$B = \frac{\mu_0 i}{2\pi a^2} r$$

$$B = \frac{\mu_0 i}{2\pi a^2} r$$

ii) $r > c$

$$B = 0$$

b) Velocity selector: A proton moves with constant velocity $v$ to the right through a region where there is a uniform magnetic field of magnitude $B$ that points into the page. There is also an electric field in this region.

\[ \bullet \rightarrow \vec{v} \quad \times \vec{B} \]

i) What is the direction of the electric field?

ii) What is the magnitude of the electric field?

\[ \Sigma \vec{F} = 0; \quad q \vec{E} = q \vec{v} \vec{B}; \quad E = \vec{v} B \]

c) How would your answer in part b) change if there were an electron instead of a proton?

i) What is the direction of the electric field?

ii) What is the magnitude of the electric field?

$E = vB$
Problem 6: (12 points)

A spherical capacitor is connected to a generator. The voltage between the capacitor’s plates is changing according to \( V(t) = V_0 \cos \omega t \). Find the displacement current through a sphere of an arbitrary radius \( r \), \( A < r < B \): radii \( A \) and \( B \) are given.

\[
\begin{align*}
\mathbf{i}_d & = \varepsilon_0 \frac{\partial}{\partial t} \int \mathbf{E} \cdot d\mathbf{s} \\
\Phi \mathbf{E} \cdot d\mathbf{s} & = \frac{Q \text{eucl}}{\varepsilon_0} \\
\Phi & = \int \mathbf{E} \cdot d\mathbf{s} = \frac{1}{\varepsilon_0} Q \\
\frac{d\Phi}{dt} & = \frac{1}{\varepsilon_0} \frac{dQ}{dt} = \frac{1}{\varepsilon_0} \frac{d}{dt} \mathcal{C} \mathbf{V} = \frac{1}{\varepsilon_0} \mathcal{C} \frac{d\mathbf{V}}{dt} = -\frac{1}{\varepsilon_0} \mathcal{C} V_0 \omega \sin \omega t \\
\mathcal{C} & = \frac{\varepsilon_0}{\varepsilon_0} \frac{Q}{\Delta \mathbf{V} \mathbf{l}}; \quad \mathcal{V}(B) - \mathcal{V}(A) = -\int_{A}^{B} \mathbf{E} \cdot d\mathbf{r}; \\
\mathbf{E} \frac{\xi \mathbf{r}}{\varepsilon_0} & = \frac{Q}{\xi \mathbf{r}}; \quad \mathbf{E} = \frac{Q}{\xi \mathbf{r}} \\
\mathcal{V}(B) - \mathcal{V}(A) & = -\int_{A}^{B} \frac{1}{\xi \mathbf{r}} \frac{Q}{\varepsilon_0} \frac{d}{d\mathbf{r}} \mathbf{r} = \frac{1}{\xi \mathbf{r}} \frac{Q}{\varepsilon_0} Q \left( \frac{1}{B} - \frac{1}{A} \right) \\
\mathcal{C} & = \frac{Q}{\Delta \mathbf{V} \mathbf{l}} = \frac{\varepsilon_0}{A - \frac{1}{B}}; \quad \mathbf{i}_d = -\frac{\varepsilon_0}{A - \frac{1}{B}} V_0 \omega \sin \omega t
\end{align*}
\]