Problem 1: (5 points)

Write Maxwell's equations in the integral form.

\[ \oint E \cdot d\hat{S} = \frac{Q_{\text{enc}}}{\varepsilon_0} \]

\[ \oint B \cdot d\hat{S} = 0 \]

\[ \oint \vec{E} \cdot d\vec{r} = -\frac{1}{\mu_0 c} \int \vec{B} \cdot d\hat{S} \]

\[ \oint \vec{B} \cdot d\vec{r} = \mu_0 i + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\hat{S} \]
Problem 2: (24 points)

a) A conducting spherical shell has inner radius A and outer radius B. It is concentric with another conducting spherical shell of inner radius C and outer radius D. The inner shell has charge +Q and the outer shell has charge -2Q.

\[ \oint \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{encl}}}{\varepsilon_0} \]

Find the electric field at

i) \( r < A \)

\[ E = 0 \quad (\text{conductor}) \]

ii) \( A < r < B \)

\[ E = 0 \quad (\text{conductor}) \]

iii) \( B < r < C \)

\[ \frac{4\pi r^2}{\varepsilon_0} E = \frac{Q}{r^2} \quad \text{radially out} \]

iv) \( C < r < D \)

\[ E = 0 \quad (\text{conductor}) \]

v) \( r > D \)

\[ \frac{4\pi r^2}{\varepsilon_0} E = \frac{Q}{r^2} \quad \text{radially in} \]

b) Sketch the electric field lines.

c) Find the difference in electric potential between \( r = A \) and \( r = \infty \): \( V(\infty) - V(A) \).

\[
(\infty) - V(A) = - \int_{A}^{\infty} E \cdot d\mathbf{r} = - \left[ \int_{A}^{B} \frac{Q}{4\pi \varepsilon_0 r^2} dr + \int_{B}^{C} \frac{Q}{4\pi \varepsilon_0 r^2} dr + \int_{C}^{D} \left( -\frac{Q}{4\pi \varepsilon_0 r^2} dr \right) \right] = \frac{Q}{4\pi \varepsilon_0} \left( \frac{1}{B} - \frac{1}{C} - \frac{1}{D} \right)
\]
d) A conducting spherical shell has inner radius A and outer radius B. It is concentric with another conducting spherical shell of inner radius C and outer radius D. The inner shell has charge +Q and the outer shell has charge -2Q. A positive charge +2Q is placed at the center of the inner shell. Find the charge density $\sigma$ at

\[
\begin{align*}
&i) \quad r = A \\
&\quad \sigma = -\frac{2Q}{4\pi A^2}
\end{align*}
\]

\[
\begin{align*}
&ii) \quad r = B \\
&\quad \sigma = \frac{3Q}{4\pi B^2}
\end{align*}
\]

\[
\begin{align*}
&iii) \quad r = C \\
&\quad \sigma = -\frac{3Q}{4\pi C^2}
\end{align*}
\]

\[
\begin{align*}
&iv) \quad r = D \\
&\quad \sigma = \frac{Q}{4\pi D^2}
\end{align*}
\]

e) A solid non-conducting sphere of radius $R$ has charge spread throughout the volume so that the charge density is

\[
\rho(r) = \rho_0 \frac{r}{R}
\]

where $\rho_0$ is a known constant, $r$ is a distance from the sphere's center. Find the electric field everywhere.

\[
\begin{align*}
E \mathbf{v} \mathbf{i} r^2 &= \frac{1}{\varepsilon_0} \int_0^r \rho_0 \frac{r}{R} v \mathbf{i} r^2 dr \\
&= \frac{1}{\varepsilon_0} \int_0^r \rho_0 \frac{r}{R} v \mathbf{i} r^2 dr \\
&= \frac{1}{\varepsilon_0} \rho_0 \frac{v^2}{4} \mathbf{i} r^2
\end{align*}
\]

Sketch the magnitude of electric field as a function of $r$. 

\[
E \quad r
\]
Problem 3: (12 points)

Consider two parallel plates of area $A$ in some circuit:

The electric field between the plates is a function of time $E = E_0 \cos \omega t$. Find the current $i$ in the wire and show that it is equal to the displacement current between the plates.

\[ i = \frac{dQ}{dt} \]

\[ \oint \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0} \]

\[ E_A = \frac{Q}{A \varepsilon_0} \]

\[ E = \frac{Q}{AE_0}, \quad Q = EA\varepsilon_0 = A\varepsilon_0 E_0 \cos \omega t \]

\[ i = \frac{dQ}{dt} = -\omega A\varepsilon_0 E_0 \sin \omega t \]

\[ i_D = \varepsilon_0 \frac{d}{dt} \oint \mathbf{E} \cdot d\mathbf{S} \]

\[ \oint \mathbf{E} \cdot d\mathbf{S} = E_0 \cos \omega t \cdot A \]

\[ i_D = \varepsilon_0 \frac{d}{dt} (A E_0 \cos \omega t) = -\omega A \varepsilon_0 E_0 \sin \omega t \]
Problem 4: (24 points)

a) In the circuit below the switch has been closed for a long time so it may be assumed that the steady state has been reached. Find the current through the resistor and the charges on the capacitor plates.

\[ i = \frac{V}{R} \]

\[ \frac{Q}{C} = 0 \Rightarrow Q = 0 \]

\[ \Phi \mathbf{E} \cdot d\mathbf{r} = 0 \]

\[ I = V - iR = 0 \]

\[ i = \frac{V}{R} \]

b) At \( t=0 \) the switch is opened. Find the charge on the capacitor as a function of time. (Assume that \( L \) includes self-inductance)

\[ \Phi \mathbf{B} \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{s} \]

\[ i = -\frac{dQ}{dt} \]

\[ \Phi = \pm li = -li \]

\[ \frac{d\Phi}{dt} = -L \frac{di}{dt} \]

\[ \frac{Q}{C} = L \frac{di}{dt} = -L \frac{d^2Q}{dt^2} \]

\[ L \frac{d^2Q}{dt^2} + \frac{1}{C} Q = 0 \]

\[ Q(t) = A \cos \omega t + B \sin \omega t \]

\[ Q(t=0) = A = 0 \]

\[ i = -\frac{dQ}{dt} = A \omega \sin \omega t - B \omega \cos \omega t \]

\[ B = -\frac{V}{R\omega} \]

\[ -L\omega^2 \sin \omega t + \frac{1}{C} B \sin \omega t = 0 \]

\[ \omega^2 = \frac{1}{LC} \]

\[ Q(t) = -\frac{V}{R\omega} \sin \omega t \]
c) In the circuit below the switch has been in the position A for a long time. Find the current through the resistor $R_2$.

\[ \Phi \mathbf{E} \cdot d\mathbf{r} = 0 \]
\[ V - i \left( R_1 + R_2 \right) = 0 \]
\[ i = \frac{V}{R_1 + R_2} \]

\[ d = L \cdot \frac{P_a}{L} \cdot t \]
\[ i(t) = \frac{V}{R_1 + R_2} \]
\[ i(t) = \frac{V}{R_1 + R_2} \cdot \mathbf{E} \cdot \frac{P_a}{L} \cdot t \]

d) At $t=0$ the switch is moved to the position B. Find the current through $R_2$, as a function of time. (Assume that $L$ includes self-inductance.)

\[ \Phi = \pm L \cdot i = -L i \]
\[ \frac{d\Phi}{dt} = -L \frac{di}{dt} \]
\[ -i_a R = L \frac{di}{dt} \]
\[ L \frac{di}{dt} + P_a i = 0 \]
\[ \frac{di}{dt} + \frac{R_2}{L} i = 0 \]
\[ i(t) = d E \cdot \frac{P_a}{L} \cdot t \]
\[ d(t) = \frac{-P_a}{L} \cdot t \]
\[ P_a = \frac{P_a}{L} \]
\[ i(t) = \frac{V}{R_1 + R_2} \cdot \mathbf{E} \cdot \frac{P_a}{L} \cdot t \]
Problem 5: (20 points)

A rectangular loop lies in the plane of the page. It has dimensions shown in the figure below. There is a long straight wire distance $D$ from the loop with current $i(t) = i_0 e^{-\beta t}$ where $i_0$ and $\beta$ are known constants. The wire from which the loop is made has resistivity $\rho$ and cross sectional area $a$.

a) Find the direction of current in the loop.

Explain your answer within this box:
CCW $i$ decreases $\Rightarrow$ $B$ decreases $\Rightarrow$
$\Phi_B$ decreases. Since Bexternal is out,
then Bself is out of page $\Rightarrow$
i is CCW

b) Calculate the current in the loop as a function of time. Ignore self-inductance.

\[
\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 i_0 \Rightarrow B = \frac{\mu_0 i_0 e^{-\beta t}}{2\pi r}, \quad R = \frac{\pi}{a} (W+H)
\]

\[
\oint \mathbf{E} \cdot d\mathbf{r} = -\frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{s}
\]

\[
\Phi = \int \mathbf{B} \cdot d\mathbf{s} = \int_{D+iH}^{D} \frac{\mu_0 i_0 e^{-\beta t}}{2\pi r} W \, dr = \frac{\mu_0 i_0 e^{-\beta t} W}{2\pi} \frac{D+iH}{D}
\]

\[
\frac{d\Phi}{dt} = \frac{\mu_0 i_0 W}{2\pi} \frac{D+iH}{D} (\beta - 1) e^{-\beta t}
\]

\[
iR = \frac{\mu_0 i_0 W B}{2\pi} \frac{D+iH}{D} e^{-\beta t}
\]

\[
i = \frac{\mu_0 i_0 W B}{2\pi} \frac{D+iH}{D} e^{-\beta t}
\]

c) Derive the equation describing the current in the loop if self-inductance of the loop is $L$. Do not solve it.

\[
\Phi_{self} = Li \quad \frac{d\Phi}{dt} = L \frac{di}{dt} \quad iR = \frac{\mu_0 i_0 W B}{2\pi} \frac{D+iH}{D} e^{-\beta t} - L \frac{di}{dt}
\]

d) Bonus (3 points) Calculate the total energy dissipated in the loop from $t = 0$ to $t = \infty$. Ignore self-inductance.

\[
\text{Energy} = \int_0^\infty i^2 R \, dt = \left( \frac{\mu_0 i_0 W B}{2\pi R} \right)^2 \frac{\mu_0 D+iH}{D} \int_0^\infty e^{-2\beta t} \, dt = \left( \frac{\mu_0 i_0 W B}{2\pi R} \right)^2 \frac{\mu_0 D+iH}{D}
\]

\[
= \frac{\mu_0 i_0 W B}{2\pi R} \frac{\mu_0 D+iH}{D}
\]
Problem 6: (20 points)

An infinitely long, hollow cylindrical wire has inner radius A and outer radius B. A current $i$ is uniformly distributed over its cross-section. a) Find the magnetic field everywhere ($r < A; A < r < B; r > B$)

\[ \begin{align*}
1) & \quad r < A \quad \oint \vec{B} \cdot d\vec{r} = \mu_0 i; \quad B = 0 \\
2) & \quad A < r < B \quad B_{\text{outer}} = \frac{\mu_0}{2\pi} \left( \frac{r^2 - A^2}{r} \right) \\
3) & \quad B_{\text{inner}} = \frac{\mu_0 i}{2\pi (B^2 - A^2)} \quad r - A^2 \quad \frac{r^2}{r} \\
\end{align*} \]

b) In addition to the hollow wire described above, there is a thin infinitely long wire positioned at a distance $d$ from the center of the hollow wire. Both wires have currents $i$ pointed into the page. Find the net magnetic field at an arbitrary point $A < x < B$ on the line connecting the centers of the wires.

Side view

\[ \begin{align*}
\vec{B} &= \vec{B}_1 + \vec{B}_2 \\
\vec{B}_1 &= \frac{\mu_0 i}{2\pi (B^2 - A^2)} \left( \frac{r^2 - A^2}{r} \right) \\
\vec{B}_2 &= \frac{\mu_0 i}{2\pi (d - x)} \\
\vec{B} &= \frac{\mu_0 i}{2\pi} \left( \frac{r^2 - A^2}{B^2 - A^2} \right) \frac{1}{r} - \frac{1}{d - x} \\
\end{align*} \]

c) There is a particle of positive charge $q$ moving with velocity of magnitude $v$ vertically up at the point located at the point $x = d/2$ in the middle of the line connecting the centers of the wires ($d/2 > B$). Find the force acting on the particle at this point.

\[ \begin{align*}
\vec{F} &= q \vec{V} \times \vec{B} \\
\vec{V} &= \vec{B}_1 + \vec{B}_2 = 0 \\
\text{or} & \quad \vec{V} \parallel \vec{B} \\
\end{align*} \]