Chapter 13

Radiation From Accelerators

One of the interesting applications of those results is to calculate the radiation one may expect from accelerators which accelerate particles to relativistic speeds. It is convenient here to expand out the $a^\mu a_\mu$ factor in

$$P = \frac{2 e^2}{3} \frac{1}{c^3} a^\mu a_\mu$$

(13.1)

One can rewrite this as

$$a^\mu = \frac{d u^\mu}{d \tau} = \frac{1}{m} \frac{d p^\mu}{d \tau}$$

(13.2)

and so

$$P = \frac{2 e^2}{3} \frac{1}{m^2 c^3} \frac{d p^\mu}{d \tau} \frac{d p_\mu}{d \tau}$$

(13.3)

one has

$$\frac{d p^\mu}{d \tau} \frac{d p_\mu}{d \tau} = (\frac{d \vec{p}}{d \tau})^2 - (\frac{d \rho}{d \tau})^2$$

(13.4)

Now

$$\frac{d \vec{p}}{d \tau} = \frac{d}{d \tau} (\gamma m \vec{v}) = \gamma \frac{d}{d t} (\gamma m \vec{v}) = \gamma m [\gamma \ddot{\vec{a}} + \vec{v} \frac{d \gamma}{d t}]$$

(13.5)
where \( \gamma = (1 - v^2/c^2)^{-1/2} \), \( \ddot{\mathbf{a}} = \frac{d}{dt} \ddot{\mathbf{v}} \), \( \frac{d\gamma}{dt} = \gamma^3 v.a/c^2 \). Hence

\[
\frac{dp}{d\tau} = \gamma^2 m [\ddot{a} + \frac{\gamma^2}{c^2} (v.a) \dot{\mathbf{v}}]
\]

(13.6)

Similarly

\[
\frac{dp^0}{d\tau} = \gamma \frac{d}{dt} \gamma m c = \gamma^4 m \frac{v.a}{c}
\]

(13.7)

Thus Eq.13.4 becomes

\[
\frac{dp^\mu}{d\tau} \frac{dp^\mu}{d\tau} = m^2 \gamma^4 [a^2 + 2 \frac{\gamma^2}{c^2} (v.a)^2 + \gamma^4 (v^2/c^2 - 1) \frac{(v.a)^2}{c^2}]
\]

(13.8)

\[
= m^2 \gamma^6 [a^2 \gamma^{-2} + \frac{(v.a)^2}{c^2}]
\]

(13.9)

which combines to

\[
= m^2 \gamma^4 [a^2 - \frac{1}{c^2} (v \times a)^2]
\]

(13.10)

Hence

\[
P = \frac{2 e^2/4\pi}{3} \frac{1}{c^3} \gamma^6 [\dot{v}^2 - \frac{1}{c^2} (v \times \dot{v})^2]
\]

(13.11)

(i) Linear Accelerator In the linear accelerator, the particle is accelerated down a straight line race track (e.g., by an electric field) Hence

\[
\dot{\mathbf{v}} \parallel \ddot{\mathbf{v}}; \quad (13.12)
\]

\[
v \times \dot{v} = 0 \quad (13.13)
\]

Thus only the first term survives

\[
P = \frac{2 e^2/4\pi}{3} \frac{1}{c^3} \gamma^6 \dot{v}^2
\]

(13.14)

We can relate this to the rate of change of energy as the particle accelerates

\[
\frac{dE}{dt} = \frac{d}{dt} \gamma mc^2 = mc^2 \gamma^3 \frac{v.a}{c^2}
\]

(13.15)
and since $\vec{v} \parallel \vec{a}$ we have where
\[
\frac{dE}{dt} = m\gamma^3 v a
\] (13.16)
or
\[
\dot{v} = \frac{1}{m\gamma^3} \frac{dE}{dt} \frac{1}{v} = \frac{1}{m\gamma^3} \frac{dE}{dx}
\] (13.17)
where
\[
\frac{dE}{dx} = \text{energy change} / \text{distance}
\] (13.18)
Hence Eq.13.14 becomes
\[
P = \frac{2}{3} \frac{\sqrt{2}}{4\pi} \frac{e^2}{m^2 c^4} \left(\frac{dE}{dx}\right)^2
\] (13.19)
Thus the power radiated is independent of the energy of the particle, but rather depends on
rate at which the particle’s energy increases over distance. If $dE/dx$ is small (a long track),
one has only a small amount of energy radiated. The above calculation has assumed that
the amount of radiation is small compared to the amount of energy being imparted to the
beam to carry out the acceleration $dE/dX$ along the beam track. Thus
\[
P << \frac{dE}{dt} = v \frac{dE}{dx} \simeq c \frac{dE}{dx}
\] (13.20)
Thus we require
\[
\frac{2}{3} \frac{\sqrt{2}}{4\pi} \frac{e^2}{m^2 c^4} \frac{dE}{dx} << 1
\] (13.21)
\[
\frac{dE}{dx} << \frac{mc^2}{\sqrt{2} \pi mc^2} = \frac{0.511\text{MeV}}{2.82 \times 10^{-13}\text{cm}} = 1.8 \times 10^{14}\frac{\text{MeV}}{m}
\] (13.22)
The r.h.s. is enormous and can always be met in practice. For example for Stanford Linear
Collider
\[
E = 50\text{GeV}, \ L = 1\text{km} = \text{track length}
\] (13.23)
so
\[
\frac{dE}{dx} = \frac{E}{L} = \frac{50 \text{GeV}}{10^3 \text{m}} = \frac{50 \text{MeV}}{m}
\] (13.24)

Thus linear collides radiate very little energy. But the same is not true for circular accelerators

(i) Circular Accelerators Here one has a magnetic field constraining the particle to move in a circle and so this case

\[ \vec{v} \perp \vec{\dot{v}}; (\vec{v} \times \vec{v})^2 = v^2 \dot{v}^2 \] (13.25)

Eq.13.11 now becomes

\[ P = \frac{2 e^2}{3} \frac{4 \pi}{c^3} \gamma^6 (1 - v^2/c^2) \dot{v}^2 \] (13.26)

or

\[ P = \frac{2 e^2}{3} \frac{4 \pi}{c^3} \gamma^4 \dot{v}^2 \] (13.27)

From Eq.9.15, 9.16

\[ \dot{v}_x = \omega_B v \cos(\omega_B t + \phi); \omega_B = \frac{ceB}{E}; \dot{v}_y = -\omega_B v \sin(\omega_B t + \phi); \dot{v}_z = 0; \dot{v}_z = 0 \] (13.28)

and so

\[ \dot{v}^2 = \dot{v}_x^2 + \dot{v}_y^2 = \omega_B^2 v^2 \] (13.29)

and using Eq. 9.20 ( i.e., \( \rho = v / \omega_B \) )

\[ \dot{v}^2 = \omega_B^4 \rho^2 \] (13.30)

The rotational frequency can be related to \( v \) by

\[ \omega_B = \frac{v}{\rho} = \frac{v}{\rho} \] (13.31)
so that

$$v^2 = \frac{v^4}{\rho^2}$$  \hspace{1cm} (13.32)

Hence Eq13.27 becomes

$$P = \frac{2}{3} \frac{e^2}{4\pi} c \left(\frac{v}{c}\right)^4 \gamma^4 \frac{1}{\rho^2}$$  \hspace{1cm} (13.33)

We can relate this to the particle energy $E = \gamma mc^2$ and so

$$P = \frac{2}{3} \left(\frac{e^2}{4\pi}\right) c \left(\frac{E}{mc^2}\right)^4 \left(\frac{v}{c}\right)^4 \frac{1}{\rho^2}$$  \hspace{1cm} (13.34)

Thus the power radiated grows like $E^4$ (Clearly if one wants to build a high energy circular accelerator, one needs to compensate by having a large ring so that the particle won’t radiate more energy than one can supply to keep the beam energy fixed at $E$. Thus the amount of energy radiated in one turn around the ring is

$$\delta E = P \frac{2\pi \rho}{v} = \frac{4\pi}{3} \frac{e^2}{4\pi} \frac{1}{\rho} \left(\frac{E}{mc^2}\right)^4 \left(\frac{v}{c}\right)^3 mc^2$$  \hspace{1cm} (13.35)

As an example the LEP2 $e^+e^-$ accelerator has parameters

**LEP** : $E = 100$GeV, $\rho = 4.24$km, $v \simeq c$

$$\delta E = \frac{4\pi}{3} \frac{2.82 \times 10^{-13} cm}{4.24 \times 10^{-5} cm} \frac{1}{0.511} \left(\frac{100 \times 10^3}{0.511}\right)^4 0.511 MeV$$  \hspace{1cm} (13.36)

$$\delta E = 2 \times 10^3$MeV/turn = 2GeV/turn  \hspace{1cm} (13.37)

Thus the limitation in building such machines is based on the practical question of how many energy one can feed into the beam on each turn to keep $E$ constant. It is clear from eq.13.35, that $\delta E \sim 1/m^4$ and so the problem is less serious for heavier particles, e.g., proton accelerators. CERN’s LHC (p-p) accelerator has the parameters

**LHC** : $E = 14$TeV, $\rho = 4.24$km, $v \simeq c$, $m_p = 938$MeV
For this case we have

\[
\delta E = \frac{4\pi}{3} 2.82 \times 10^{-13} cm \cdot \frac{0.511/938}{4.24 \times 10^5 cm} \left( \frac{14 \times 10^6}{938} \right)^4 938 MeV
\]  

(13.38)

\[
\delta E = 7.4 \times 10^{-2} MeV/\text{turn}
\]  

(13.39)

which is much smaller due to the higher mass. However it is harder to feed energy to a proton for the same reason and so radiation plays an important role in the design of the LHC.