The Sun: Our Nearest Star

Check out:
http://www.boston.com/bigpicture/2008/10/the_sun.html
http://sohowww.nascom.nasa.gov/gallery/
http://www.youtube.com/watch?v=JeAmKKrIvIc
http://www.youtube.com/watch?v=YJBrMXSn-hU
Earth’s Magnetic Field
Earth’s Magnetic Field
Earth’s Magnetic Field

Magnetic North Pole keeps moving.
Location from 1831-2001:
Earth’s Magnetic Field Reverses over time

- Normal Magnetic polarity
- Reversed polarity

Bands correspond to ~300,000 yr. Last one was ~780,000 yr ago....
Solar Magnetic Dynamo

equator

differential rotation

magnetic field line

N

S

time

time

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The Sun: Our Nearest Star

Check out:
http://www.boston.com/bigpicture/2008/10/the_sun.html
http://sohowww.nascom.nasa.gov/gallery/
Solar Magnetic Dynamo

The Magnetic Butterfly Diagram
average magnetic fields at the Sun's surface

Legend
+40 gauss
+20 gauss
0 gauss
-20 gauss
-40 gauss

Magnetic South
Magnetic North

Magnetic North
Magnetic South
Gravitational Force Pulls on Everything. In Stars, there must be some Pressure to balance Gravity.

Consider a cylinder of mass, \( dm \), located a distance \( r \) from center of a spherical star. Top of the cylinder has an area, \( A \), and it has a height, \( dr \). Net force on the Cylinder is:

\[
dm \frac{d^2 r}{dt^2} = F_g + F_{P,t} + F_{P,b}
\]

Write \( F_{R,t} \) in terms of \( F_{R,b} \) and extra force \( dF_P \)

\[
F_{P,t} = -(F_{P,b} + dF_P)
\]

Substitution Gives:

\[
dm \frac{d^2 r}{dt^2} = F_g - dF_P
\]
The gravitational force is: $F_g = -\frac{GM_r}{r^2}$. The Pressure is $P = F / A$, or $dF_P = A \, dP$. This Yields:

$$dm \frac{d^2 r}{dt^2} = -G \frac{M_r \, dm}{r^2} - A \, dP$$

Rewrite $dm$ in terms of the density, $\rho$, which gives $dm = \rho \, A \, dr$.

$$\rho A dr \, \frac{d^2 r}{dt^2} = -G \frac{M_r \rho \, A \, dr}{r^2} - A \, dP$$

Divide by the Area, $A$, and assume star is static ($\frac{d^2 r}{dt^2} = 0$), and rearrange terms:

$$\frac{dP}{dr} = -G \frac{M_r \rho}{r^2} = -\rho \, g$$

Hydrostatic Equilibrium
Physics of Stellar Interiors

Where does the Pressure come from?

Need to derive the equation of state. For example, pressure equation of state for the “ideal” gas law is \( PV = N k T \).

Consider another cylinder of length \( \Delta x \) and area \( A \). Each gas particle has mass \( m \) and interacts via collisions only. The impulse \( (f \Delta t) \) is the negative of the change in momentum, \( f \Delta t = -\Delta p = 2p_x \).
Physics of Stellar Interiors

\[ \mathbf{f} \Delta t = -\Delta \mathbf{p} = 2p_x \mathbf{i} \]

Time interval between “collisions” is \( \Delta t = 2 \Delta x / v_x \). This produces an effective force of \( f = 2p_x / \Delta t = p_x v_x / \Delta x. = m v_x^2 / \Delta x. \)

Note: \( v^2 = v_x^2 + v_y^2 + v_z^2 \). For large volumes, the average velocity in each direction should be equal, \( \langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle = v^2 / 3. \) Substituting, \( pv/3 \) for \( p_x v_x \) gives force per average particle:

\[ f(p) = \frac{1}{3} \frac{pv}{\Delta x} \]
Physics of Stellar Interiors

\[ f(p) = \frac{1}{3} \frac{pv}{\Delta x} \]

Usually particles have a range of momenta, total number of particles is

\[ N = \int_{0}^{\infty} N_p \, dp \]

Then total force from all particles is

\[ dF(p) = f(p) N_p \, dp = \frac{1}{3} \frac{N_p}{\Delta x} pv \, dp. \]

Integrating over all momenta gives

\[ F = \frac{1}{3} \int_{0}^{\infty} \frac{N_p}{\Delta x} pv \, dp \]

Divide both sides by the Area, A.

Rewrite number of particles as number per Volume, \( n_p = \frac{N_p}{\Delta V} \).

\[ P = \frac{1}{3} \int_{0}^{\infty} n_p pv \, dp \]

Pressure Integral
Physics of Stellar Interiors

\[ P = \frac{1}{3} \int_{0}^{\infty} n_p p v^2 dp \]

Rewrite \( p = mv \), which gives

\[ P = \frac{1}{3} \int_{0}^{\infty} m n_v v^2 dp \]

\( n_v \ dv \) for an ideal gas is the Maxwell Boltzmann distribution.

Plugging in an evaluating integral gives, \( P_g = nkT \), where \( n = \int_{0}^{\infty} n_v dv \)

Express the ideal gas law using the average mass density of particles of different masses, where \( m \) is the average mass of a gas particle.

\[ n = \frac{\rho}{\bar{m}} \]

Pressure becomes:

\[ P_g = \frac{\rho kT}{\bar{m}} \]

Define **mean molecular weight**, \( \mu \equiv \frac{\bar{m}}{m_H} \)

Mean molecular weight is the average mass of a free particle in the gas, in units of the mass of hydrogen.

For example, for a gas that is 10% He and 90% H by number, then \( \mu \approx (0.1 \times 4m_p + 0.9 \times m_p) / m_H = 1.3 \).
Physics of Stellar Interiors

Therefore, rewrite gas law as

\[ P_g = \frac{\rho k T}{\mu m_H} \]

Mean molecular weight depends on the composition (% of different atoms) and ionization state because free electrons must be counted. Complete treatment needs to use Saha Equation to account for everything.

We will consider two cases (1) gas all neutral and (2) gas all ionized.

(1) NEUTRAL GAS:

\[ \bar{m}_n = \frac{\sum_j N_j m_j}{\sum_j N_j} \]

\[ \mu_n = \frac{\sum_j N_j A_j}{\sum_j N_j} \]

m\textsubscript{j} and N\textsubscript{j} are the mass and # of atoms of type j.

divide by m\textsubscript{H}, let A\textsubscript{j} = m\textsubscript{j} / m\textsubscript{H}:
Physics of Stellar Interiors

(1) NEUTRAL GAS:
\[ \mu_n = \frac{\sum_j N_j A_j}{\sum_j N_j} \]

(2) IONIZED GAS:
\[ \mu_i \approx \frac{\sum_j N_j A_j}{\sum_j N_j (1 + z_j)} \]

Where \( z_j \) accounts for the nuclei and \# of free electrons.

Invert expressions \((1/\mu)\), in terms of mass fractions

\[ \frac{1}{\mu_n m_H} = \frac{\sum_j N_j}{\sum_j N_j m_j} = \frac{\text{total \# of particles}}{\text{total mass of gas}} \]

\[ = \sum_j \frac{\# \text{ of particles from } j}{\text{mass of particles from } j} \times \frac{\text{mass of particles from } j}{\text{total mass of gas}} \]

\[ = \sum_j \frac{N_j}{N_j A_j m_H} X_j = \sum_j \frac{X_j}{A_j m_H} \]

\( X_j \) is the mass fraction of atoms of type \( j \).
Multiply both sides by $m_H$ and we get for neutral gas:

$$\frac{1}{\mu_n} = \sum_j \frac{X_j}{A_j}$$

Now define **mass fraction**, the fractional abundance (by mass) of an element. Fraction of hydrogen is $X$. Fraction of Helium is $Y$, Fraction of everything else is $Z$.

$$X = \frac{\text{total mass of H}}{\text{total Mass}}$$
$$Y = \frac{\text{total mass of He}}{\text{total Mass}}$$
$$Z = \frac{\text{total mass of Li through Uuo}}{\text{total Mass}}$$

And $X + Y + Z = 1$

For neutral gas, we have:

$$\frac{1}{\mu_n} \approx X + \frac{1}{4} Y + \langle 1/A \rangle_n Z$$
Physics of Stellar Interiors

For ionized gas, include electron in calculation. Ionized hydrogen gives 1 electron, He gives 2, etc. This gives (where $z_j$ is the atomic # of element $j$).

$$\frac{1}{\mu_i} = \sum_j \frac{1 + z_j}{A_j} X_j$$

Including H and He explicitly and averaging over everything else:

$$\frac{1}{\mu_i} \approx 2X + \frac{3}{4} Y + \langle (1 + z)/A \rangle_i Z$$

For elements much heavier than He, $1 + z_j \approx z_j$, because $z_j \gg 1$. Also holds that $A_j \approx 2z_j$ because sufficiently massive atoms have roughly the same number of protons and neutrons in their nuclei. Therefore,

$$\langle (1 + z)/A \rangle_i \approx \frac{1}{2}$$

For $X=0.70, Y=0.28, \text{ and } Z=0.02$ (the mass fractions of the Sun), we get $\mu_n = 1.30$ and $\mu_i = 0.62$. 

Average Kinetic Energy Per Particle

We can derive the average kinetic energy of a particle from the ideal gas law. We had previously,

\[ nkT = \frac{1}{3} \int_0^\infty m n_v v^2 \, dv \]

Rewriting, we get

\[ \frac{1}{n} \int_0^\infty n_v v^2 \, dv = \frac{3kT}{m} \]

The left hand side is the average velocity squared given the Maxwell-Boltzmann distribution, so this integral is just \( <nv^2> \). This gives:

\[ \bar{v}^2 = \frac{3kT}{m} \]

Which implies:

\[ \frac{1}{2} m \bar{v}^2 = \frac{3}{2} kT. \]

Note that the factor of 3 came about because we averaged over 3 coordinate directions (3 degrees of freedom). The average kinetic energy of a particle is \( kT/2 \) for each degree of freedom in the system.
Fermi-Dirac and Bose-Einstein Statistics

So far we have ignored the effects of Quantum Mechanics. In some cases we must consider them.

**Fermi-Dirac** statistics dictate the properties of 1/2 integer spin particles, such as in white dwarf stars (made of electrons, protons, and neutrons) and neutron stars (made of neutrons). These are all **fermions**.

**Bose-Einstein** Statistics dictate properties of integer spin particles (such as photons, **bosons**).

Limits at low densities give classical results that we have so-far derived.
Radiation Pressure

Photons carry momentum, \( p = h\nu/c \). They are capable of delivering the “impluse” to gas particles (they are absorbed, reflected and scattered, which transfers momentum into the gas atoms/ions).

Rewrite our Pressure Integral in terms of photon momenta. Using \( n_p \, dp = n_\nu \, d\nu \) in our pressure integral, we have

\[
P_{\text{rad}} = \frac{1}{3} \int_{0}^{\infty} h\nu \, n_\nu \, d\nu
\]

One can solve this using Bose-Einstein statistics for Photons (which are Bosons), or realize that \((h\nu \, n_\nu)\) is the energy density distribution of photons. Or,

\[
P_{\text{rad}} = \frac{1}{3} \int_{0}^{\infty} u_\nu \, d\nu
\]

For Blackbody radiation, we can solve for \( P_{\text{rad}} \) using the energy density formula we derived way back in week 3.

\[
u_\nu \, d\nu = \frac{8\pi h\nu^3/c^3}{e^{h\nu/kT} - 1} \, d\nu
\]

which gives

\[
P_{\text{rad}} = \frac{1}{3} aT^4
\]

(where \( a = 4\pi/c \))
Total Pressure

Now we can combine our Pressure terms for stars, combining the thermal pressure for particles of a temperature $T$ and the contribution from the photon pressure.

$$P_t = \frac{\rho k T}{\mu m_H} + \frac{1}{3} aT^4$$
Example: Pressure and Temperature in Sun

We can estimate the pressure and temperature at the center of the Sun. Use $M_r = 1\ M_\odot$, $r = 1\ R_\odot$, $\rho = \rho_\odot = 1410\ \text{kg m}^{-3}$ (average solar density).

Assume also that the surface pressure is $P_S = 0$.

$$\frac{dP}{dr} \approx \frac{(P_S - P_C)/(R_S - 0)}{R_\odot} = -\frac{P_C}{R_\odot}$$

Inserting this into our hydrostatic equilibrium equation:

$$\frac{dP}{dr} = -G \frac{M_r \rho}{r^2}$$

$$P_C \approx G \frac{M_\odot \rho_\odot}{R_\odot} = 2.7 \times 10^{14}\ \text{N m}^{-2}$$

For more accurate value, integrate the hydrostatic equilibrium equation:

$$\int_{P_S}^{P_C} dP = P_C = -\int_{R_S}^{R_\odot} \frac{GM_r \rho}{r^2} dr$$

This gives a central pressure of $P_C = 2.34 \times 10^{16}\ \text{N m}^{-2}$.

$1\ \text{atm} = 10^5\ \text{N m}^{-2}$, so center of the Sun has a pressure of $2.3 \times 10^{11}\ \text{atm}!$
Example: Pressure and Temperature in Sun

For Temperature, neglect the radiation pressure and we have:

\[ P_g = \frac{\rho k T}{\mu m_H} \]

Rewriting, we have \[ T_C = \frac{P_C \mu m_H}{\rho k} \]

Using \( \rho_\odot = 1410 \text{ kg m}^{-3} \) (average solar density) and \( \mu_i = 0.62 \), and the estimated value for the pressure (last slide) we have \( T_C \approx 1.44 \times 10^7 \text{ K} \).

For fun, you can calculate the radiation pressure, which is

\[ P_{\text{rad}} = \frac{1}{3} a T^4 = 1.53 \times 10^{13} \text{ N m}^{-2}, \]

or 0.065\% that of the gas pressure.
Stellar Energy Sources

Gravitational Energy
Recall that Gravitational Potential Energy is $U = -G \frac{Mm}{r}$. This says that as particles move toward each other, the potential energy is more negative.

For two particles may eventually end up in a bound orbit. The Virial Theorem says that the total energy in the bound system is $E = (1/2)U$. So, 50% of the energy is available to be radiated away.

\[
dF_{g,i} = G \frac{M_r dm_i}{r^2}
\]

\[
dU_{g,i} = -G \frac{M_r dm_i}{r}
\]
Stellar Energy Sources

Gravitational Energy

Now, instead of a point mass, $dm_i$, consider a shell of thickness $dr$ with mass $dm = 4\pi r^2 \rho \, dr$

Differential potential energy is

$$dU_g = -G \frac{M_r 4\pi r^2 \rho}{r} \, dr$$

Integrating gives Potential Energy

$$U_g = -4\pi G \int_0^R M_r \rho r \, dr$$

Approximate density as constant over volume

$$\rho \approx \bar{\rho} = \frac{M}{(4/3 \pi R^3)}$$

$$M_r \approx \frac{4}{3} \pi r^3 \bar{\rho}$$
Stellar Energy Sources

Gravitational Energy

Inserting these into the Potential Energy equation gives:

\[ U_g = -4\pi G \int_0^R \left( \frac{4}{3} \pi r^3 \bar{\rho} \right) \bar{\rho} r \, dr \]

\[ = -\frac{16}{3} \pi^2 G \bar{\rho}^2 \int_0^R r^4 \, dr \]

Solving gives:

\[ U_g \approx -\frac{16\pi^2}{15} G \bar{\rho}^2 R^5 \approx \frac{3}{5} \frac{G M^2}{R} \]

Applying the Virial Theorem, \( E = U/2 \):

\[ E \approx -\frac{3}{10} \frac{G M^2}{R} \]

This is the amount of energy “Lost” in the gravitational collapse of system that ends up in a bound state.
Stellar Energy Sources

Gravitational Energy

Example, assume the Sun started as a spherical cloud of Hydrogen with a very, very large Radius, $R_i \gg R_\odot$. The energy radiated away is

$$\Delta E_g = - (E_f - E_i) \approx -E_f \approx \frac{3}{10} \frac{G M_\odot^2}{R_\odot} = 1.1 \times 10^{41} \text{ J}.$$ 

We know the Sun has a Luminosity of $3.826 \times 10^{26} \text{ W (J/s)}$. Dividing $\Delta E_g$ by the luminosity gives an estimate for how long it would take the Sun to radiate away its gravitational potential energy.

$$t = \frac{\Delta E_g}{L_\odot} \approx 10^7 \text{ yr}.$$ 

Called the “Kelvin-Helmholtz” timescale (people who first worked it out).

Does this make sense?
Most Abundant Elements in the Solar Photosphere.

Most abundant cosmic elements are H, He, O, C, Ne, N, Mg, Si, Fe. True for cosmos and the Sun.

<table>
<thead>
<tr>
<th>Element</th>
<th>Atomic #</th>
<th>Log Relative Abundance</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1</td>
<td>12.00</td>
</tr>
<tr>
<td>He</td>
<td>2</td>
<td>10.93 ± 0.004</td>
</tr>
<tr>
<td>O</td>
<td>8</td>
<td>8.83 ± 0.06</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>8.52 ± 0.06</td>
</tr>
<tr>
<td>Ne</td>
<td>10</td>
<td>8.08 ± 0.06</td>
</tr>
<tr>
<td>N</td>
<td>7</td>
<td>7.92 ± 0.06</td>
</tr>
<tr>
<td>Mg</td>
<td>12</td>
<td>7.58 ± 0.05</td>
</tr>
<tr>
<td>Si</td>
<td>14</td>
<td>7.55 ± 0.05</td>
</tr>
<tr>
<td>Fe</td>
<td>26</td>
<td>7.50 ± 0.05</td>
</tr>
<tr>
<td>S</td>
<td>16</td>
<td>7.33 ± 0.11</td>
</tr>
<tr>
<td>Al</td>
<td>13</td>
<td>6.47 ± 0.07</td>
</tr>
<tr>
<td>Ar</td>
<td>18</td>
<td>6.40 ± 0.06</td>
</tr>
<tr>
<td>Ca</td>
<td>20</td>
<td>6.36 ± 0.02</td>
</tr>
<tr>
<td>Ng</td>
<td>11</td>
<td>6.33 ± 0.03</td>
</tr>
<tr>
<td>Ni</td>
<td>28</td>
<td>6.25 ± 0.04</td>
</tr>
</tbody>
</table>
Stellar Energy Sources

Nuclear Energy

An Element is specified by a # of protons, Z.

How do protons stay together in nuclei?

Turns out, then need neutrons to glue the nuclei together or the +e charge of the protons would break the nuclei apart.

An Isotope of an element is identified by the # of neutrons, N, in a nucleus.

The number of nucleons (protons + neutrons) is \( A = Z + N \).

A is refereed to as the mass number. Masses of atomic particles are:

\[
\begin{align*}
m_p &= 1.67262158 \times 10^{-27} \text{ kg} = 1.00727646688 \text{ u} \\
m_n &= 1.67492716 \times 10^{-27} \text{ kg} = 1.00866491578 \text{ u} \\
m_e &= 9.10938188 \times 10^{-31} \text{ kg} = 0.0005485799110 \text{ u}
\end{align*}
\]

1 u is the mass of a Carbon-12 nucleus ÷ 12 (definition).

But, a carbon 12 nucleus might have a mass of \( 6m_p + 6m_n = 12.096 \text{ u} \).

Why is this > 12 u ?!!!
Stellar Energy Sources

Nuclear Energy

Nuclear Fusion releases energy. It converts mass into energy. Recall Relativity, \( E=mc^2 \). 1 u = 931.494013 MeV/c^2.

Note that the mass of hydrogen in the ground state, \( m_H = 1.00782503214 \text{ u} \). This says that \( m_H < m_p + m_e = 1.00783 \). The difference is actually -13.6 eV.

The Sun is fusing He from H. A He-4 nucleus has a mass of 4.0026 u.

4 Hydrogen atoms have a mass of 4.0313 u.

\( \Delta m = 0.028697 \text{ u}, \) or 0.7% of the total energy.

This is an energy of \( E=\Delta mc^2 = 26.731 \text{ MeV} \). This is the binding energy of a He-4 nucleus. To break apart a He-4 nucleus takes this much energy.
INTERNATIONAL UNION OF PURE AND APPLIED CHEMISTRY

INORGANIC CHEMISTRY DIVISION
COMMISSION ON ATOMIC WEIGHTS AND ISOTOPIC ABUNDANCES

ATOMIC WEIGHTS OF THE ELEMENTS:
REVIEW 2000

(IUPAC Technical Report)

Prepared for publication by
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Stellar Energy Sources

Nuclear Energy

Example: How much nuclear energy is available in the Sun.
Assume the Sun was 100% Hydrogen initially and that only 10% of the inner mass is involved in fusion (gas in the nucleus of the Sun).

0.7% of the hydrogen mass is converted to energy, so:

\[ E_{\text{nuclear}} = 0.1 \times 0.007 \times M_\odot c^2 = 1.3 \times 10^{44} \text{ J} \]

(Recall that for gravity, \( \Delta E_g = 1.1 \times 10^{41} \text{ J} \).)

Similarly, we can derive the timescale using the Solar Luminosity:

\[ t = \Delta E_{\text{nuclear}} / L_\odot \approx 10^{10} \text{ yr.} \]
Stellar Energy Sources

Nuclear Energy

Nuclear Fusion is governed by strong nuclear force. But, to fuse two hydrogen atoms, they must overcome the Coulomb barrier. Recall that the electric (repulsive!) force between two protons is \((1/4\pi\varepsilon_0) q^2/r^2\).
Stellar Energy Sources

Nuclear Energy

Make assumption that the energy to overcome the Coulomb barrier is the thermal energy of the gas. Refer to relative velocity of two nuclei (protons) using their reduced mass, $\mu_m$.

The point where the total energy is zero is where the two nuclei get to their closest approach and then are repulsed by the Coulomb force:

$$\frac{1}{2}\mu_m v^2 = \frac{3}{2}kT = (1/\pi \varepsilon_0) (Z_1 Z_2 e^2)/r$$

$$T = (Z_1 Z_2 e^2) / 6\pi \varepsilon_0 kr$$

For two hydrogen atoms, $Z_1=Z_2=1$ and $r\approx1$ fm = $10^{-15}$ m. This then gives $T \approx 10^{10}$ K.

But, the center of the Sun is only $\sim 1.6 \times 10^7$ K.

**How does nuclear fusion take place?**
Stellar Energy Sources

Nuclear Energy

Answer is **Quantum Mechanics**. Recall that we learned that the Heisenberg Uncertainty principle states that the uncertainties in the position and momentum of a particle are related by

\[ \Delta x \Delta p > \frac{\hbar}{2} \]

Quantum Mechanics allows that the location can not be known precisely. Allows particle to “exist” past Coulomb barrier, and eventually fuse!
Stellar Energy Sources

Nuclear Energy

Estimate quantum mechanical effect on temperature. Use that the “wavelength” of a particle is $\lambda = \frac{h}{p}$, which means we can rewrite the Kinetic energy as:

$$(1/2)\mu m v^2 = \frac{p^2}{2\mu m} = \frac{(h/\lambda)^2}{2\mu m}$$

Then we can set the distance of closest approach equal to one wavelength (where the height of the potential barrier is equal to the Kinetic Energy).

$$(1/4\pi\varepsilon_0) Z_1 Z_2 e^2 / \lambda = \frac{(h/\lambda)^2}{2\mu m}$$

Solving for $\lambda$ and substituting $r=\lambda$ into

$$T = (Z_1 Z_2 e^2) / 6\pi\varepsilon_0 kr$$

we have

$$T_{quantum} = Z_1 Z_2 e^4 \mu_m / (12\pi^2\varepsilon_0^2h^2k)$$

For two protons, $\mu_m = m_p/2$ and $Z_1=Z_2=1$ which gives $T_{quantum} \approx 10^7$ K.

That’s better, and quantum mechanics matters!
Stellar Nucleosynthesis

Conservation laws

There certain conservation laws in nature, that must be obeyed. Some are

Conservation of electric charge: all products have same net charge as reactants.

Conservation of Lepton number: Leptons are “light things”, electrons, positrons, muons, neutrinos. Must have same lepton # before and after a reaction.

Example: \( e^- + e^+ \rightarrow 2\gamma \)

On LHS an electron and positron annihilate.

Initial total charge is \( q_i = -1 + 1 = 0 \).

Total lepton # is \( L_i = 1 \) (for \( e^- \)) + -1 (for \( e^+ \)) = 0.

Final charge \( q_f = 0 \) and final lepton # \( L_f = 0 \).

Two photons are required to conserve momentum (another conservation law).
Stellar Nucleosynthesis

Fusion Reactions in Stars
There are several paths that Stars can use to fuse He from H.

The Proton-Proton (PP) chain

\[ 4 \ \text{H} \rightarrow ^{4}\text{He} + 2e^+ + 2\nu_e + 2\gamma \]

The intermediate steps involve the intermediate products Deuterium ($^2\text{H}$) and helium-3 ($^3\text{He}$)

\[
\begin{align*}
\text{H} + \text{H} & \rightarrow ^2\text{H} + e^+ + \nu_e \\
\text{H} + ^2\text{H} & \rightarrow ^3\text{He} + \gamma \\
^{3/2}\text{He} + ^3\text{He} & \rightarrow ^4\text{He} + 2 \ \text{H}
\end{align*}
\]

Each reaction has its own rate because each has its own Coulomb Barrier to overcome.
Stellar Nucleosynthesis
Fusion Reactions in Stars

A variant is the PP II chain (starting with step 3 in the PP)

\[
\begin{align*}
\frac{3}{2}\text{He} + \frac{4}{2}\text{He} & \rightarrow \frac{7}{2}\text{Be} + \gamma \\
\frac{7}{2}\text{Be} + e^- & \rightarrow \frac{7}{3}\text{Li} + \nu_e \\
\frac{7}{3}\text{Li} + \frac{1}{1}\text{H} & \rightarrow 2 \frac{4}{2}\text{He}
\end{align*}
\]

Yet another variant is the PP III chain (starting with Step 2 in PPII)

\[
\begin{align*}
\frac{7}{2}\text{Be} + \frac{1}{1}\text{H} & \rightarrow \frac{8}{3}\text{B} + \gamma \\
\frac{8}{3}\text{B} & \rightarrow \frac{8}{4}\text{Be} + e^+ + \nu_e \\
\frac{8}{4}\text{Be} & \rightarrow 2 \frac{4}{2}\text{He}
\end{align*}
\]

There are other possibilities. You can imagine others, then you work out what the timescales are (how long does the reaction take). This determines if it matters.
Stellar Nucleosynthesis
Fusion Reactions in Stars

\[ ^1_1H + ^1_1H \rightarrow ^2_1H + e^+ + \nu_e \]

\[ ^2_1H + ^1_1H \rightarrow ^3_2He + \gamma \]

\[ ^3_2He + ^3_2He \rightarrow ^4_2He + 2 ^1_1H \] (PP I)

69% 31%

\[ ^3_2He + ^4_2He \rightarrow ^7_4Be + \gamma \]

99.7% 0.3%

\[ ^7_4Be + e^- \rightarrow ^7_3Li + \nu_e \]

\[ ^7_3Li + ^1_1H \rightarrow 2 ^4_2He \] (PP II)

\[ ^7_4Be + ^1_1H \rightarrow ^8_3B + \gamma \]

\[ ^5_3B \rightarrow ^8_4Be + e^+ + \nu_e \]

\[ ^8_4Be \rightarrow 2 ^4_2He \] (PP III)
Stellar Nucleosynthesis
Fusion Reactions in Stars

The Carbon-Nitrogen-Oxygen (CNO) Cycle

This cycle uses C, N, and O as a catalyst for the fusion of He. Proposed by Hans Bethe in 1938. There are several variants.

Primary CNO Cycle

\[
\begin{align*}
^{12}\text{C} + ^1\text{H} &\rightarrow ^{13}\text{N} + \gamma \\
^{13}\text{N} &\rightarrow ^{13}\text{C} + e^+ + \nu_e \\
^{13}\text{C} + ^1\text{H} &\rightarrow ^{14}\text{N} + \gamma \\
^{14}\text{N} + ^1\text{H} &\rightarrow ^{15}\text{O} + \gamma \\
^{15}\text{O} &\rightarrow ^{15}\text{N} + e^+ + \nu_e \\
^{15}\text{N} + ^1\text{H} &\rightarrow ^{16}\text{C} + ^4\text{He}
\end{align*}
\]

Variant CNO Cycle (occurs 0.04% of the time)

Starting with the last line of the Primary CNO cycle

\[
\begin{align*}
^{15}\text{N} + ^1\text{H} &\rightarrow ^{16}\text{O} + \gamma \\
^{16}\text{O} + ^1\text{H} &\rightarrow ^{17}\text{F} + \gamma \\
^{17}\text{F} &\rightarrow ^{17}\text{O} + e^+ + \nu_e \\
^{17}\text{O} + ^1\text{H} &\rightarrow ^{14}\text{N} + ^4\text{He}
\end{align*}
\]
Stellar Nucleosynthesis
Fusion Reactions in Stars

Note that as H → He the mean molecular weight increases. The ideal gas law predicts that the central pressure, $P_C$, will then decrease. The star will no longer be in equilibrium and will collapse, raising the temperature. At some point He begins to “burn” (fuse to C).

**Triple Alpha Process**

Fusion of Three $^4$He nuclei to form $^{12}$C. Called “alpha” because an alpha particle is a $^4$He nucleus.

\[
\begin{align*}
^4\text{He} + ^4\text{He} &\rightarrow ^8\text{Be} \\
^8\text{Be} + ^4\text{He} &\rightarrow ^{12}\text{C} + \gamma
\end{align*}
\]
Stellar Nucleosynthesis

Fusion Reactions in Stars

Carbon and Oxygen Burning

After sufficient Carbon has been produced, further fusion occurs. Generally this is done by adding $^4_2\text{He}$ nuclei. These elements are called “α” elements (created by α-particle capture).

$$^{12}_6\text{C} + ^4_2\text{He} \rightarrow ^{16}_8\text{O} + \gamma$$  $$^{16}_8\text{O} + ^4_2\text{He} \rightarrow ^{20}_8\text{Ne} + \gamma$$

At higher temperatures still:

$$^{12}_6\text{C} + ^{12}_6\text{C} \rightarrow \begin{cases} ^{16}_8\text{O} + 2^4_2\text{He} \quad \text{***} \\ ^{20}_{10}\text{Ne} + ^4_2\text{He} \\ ^{23}_{11}\text{Na} + ^1_1\text{H} \\ ^{23}_{12}\text{Mg} + n \quad \text{***} \\ ^{24}_{12}\text{Mg} + \gamma \end{cases}$$

$$^{16}_8\text{O} + ^{16}_8\text{O} \rightarrow \begin{cases} ^{24}_{12}\text{Mg} + 2^4_2\text{He} \quad \text{***} \\ ^{28}_{18}\text{Si} + ^4_2\text{He} \\ ^{31}_{15}\text{P} + ^1_1\text{H} \\ ^{31}_{16}\text{S} + n \\ ^{32}_{16}\text{S} + \gamma \end{cases}$$

*** reactions are endothermic - they absorb energy rather than release it. These are rare.
Binding energy is the amount of energy that was released in the creation of an element. Recall that the fusing of 4 Hydrogen atoms into Helium releases $\Delta m = 0.028697 \text{u}$, or an energy of $E = \Delta mc^2 = 26.731 \text{ MeV}$. 

$$E_b = \Delta mc^2 = [Z m_p + (A - Z) m_n - m_{\text{nucleus}}] c^2$$
Most Abundant Elements in the Solar Photosphere.

Most abundant cosmic elements are H, He, O, C, Ne, N, Mg, Si, Fe. True for cosmos and the Sun.

<table>
<thead>
<tr>
<th>Element</th>
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<th>Log Relative Abundance</th>
</tr>
</thead>
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<tr>
<td>H</td>
<td>1</td>
<td>12.00</td>
</tr>
<tr>
<td>He</td>
<td>2</td>
<td>10.93 ± 0.004</td>
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Note: These are all α elements. Stars are very efficient at making α elements!
Stellar Nucleosynthesis
Fusion Reactions in Stars

Where do elements with \( Z > 26 \) come from?

\textbf{s-Process Nucleosynthesis}

One way is by s-process, s- for “slow”. Free neutrons do not feel Coulomb Barrier and collide with nuclei. Occasionally they stick, making larger nuclei. If neutron flux is not too great, these heavier nuclei decay before more neutron captures.

Technetium (Tc) has no stable isotopes (all decay). But it is found in the atmospheres of giant stars. Most abundant isotope, \(^{99}\text{Tc}\) has a half-life of 200,000 yrs, much less than lifetime of star. Must be forged in the star and dredged up.

However, there are occasions when the neutron flux is much, much higher... especially when nucleosynthesis stops in stars, causing the cores to collapse, which increases the neutron density.
Nature of Stars

Hydrogen makes up 70% of stars ($X \sim 0.7$) and heavy metals a small fraction ($0 < Z < 0.03$)

Assuming that a star forms from a gas cloud that is homogeneous in its heavy metal distribution (a safe assumption), then all stars should start off using the pp chain or CNO cycle to convert H to He.

During nucleosynthesis, the surface of the star is not completely static, the observational characteristic of the star must change as a consequence of the central nuclear reactions.

But, these changes are slow ($\sim 10^6$-10$^9$ yrs), and so are the evolutionary stages of stars.

Recall that: $P_c \approx G M \rho / R$ and $T_c = \frac{P_c \mu m_H}{\rho k}$

Therefore, as the mass increases, so does the central pressure and temperature.
Nature of Stars

Because nuclear reaction rates depend on temperature, the dominate processes depend on mass. In lower mass stars, the pp-chain dominates. In higher-mass stars (with enough heavy metals), the CNO process will dominate.

For very low mass stars, the central temperature will diminish to the point where no nuclear fusion occurs. For a star with Solar compositions this is 0.072 M⊙.

For very high mass stars, (>90 M⊙), the radiation pressure can create thermal oscillations that produce variations in their luminosity on 8 hour timescales.

\[ P_t = \frac{\rho kT}{\mu m_H} + \frac{1}{3} aT^4 \]
Nature of Stars

Very massive stars have high luminosities resulting from very high central temperatures. Radiation pressure can dominate! Rewrite the pressure gradient:

\[ \frac{dP}{dr} \approx -\left( \frac{\kappa \rho}{c} \right) \frac{L}{(4\pi r^2)} \]

Where \( \kappa \) is a constant (opacity) that measures how readily the material (gas) absorbs the light.

Hydrostatic equilibrium requires:

\[ \frac{dP}{dr} = -G \frac{M_r \rho}{r^2} \]

Combining these equations and solving for \( L \) gives the maximum luminosity for a star such that it can remain in hydrostatic equilibrium:

\[ L_{edd} = \left( \frac{4\pi G c}{\kappa} \right) M \]

This is the Eddington Limit (named for Arthur Eddington). It appears in many astrophysical applications.

Arthur Eddington (1882-1944)
Nature of Stars

This is the Eddington Limit (named for Arthur Eddington). It appears in many astrophysical applications.

\[ L_{\text{edd}} = \left[ \frac{4\pi G c}{\kappa} \right] M \]

For high mass stars (highest temperatures, luminosities), \(T \sim 50,000\) K. Most hydrogen is ionized in these stars’ atmospheres.

Opacity comes from interactions between photons and electrons.

\[ \kappa = 0.02(1+X) \text{ m}^2 \text{ kg}^{-1} \], where \(X\) is the hydrogen mass fraction.

For a 90 M\(_{\odot}\) we find that \(L_{\text{edd}} = 3.5 \times 10^6\) L\(_{\odot}\).

The expected luminosity for such a star’s temperature is \(L \sim 10^6\) L\(_{\odot}\), 3x less than the Eddington limit.
Nature of Stars

Theoretical HR Diagram

Main-sequence

$L \sim M^4$

$\log_{10} (L/L_\odot)$ vs $\log_{10} T_e (K)$

Points:
- $120 M_\odot$
- $60 M_\odot$
- $25 M_\odot$
- $15 M_\odot$
- $9 M_\odot$
- $5 M_\odot$
- $3 M_\odot$
- $2 M_\odot$
- $1.5 M_\odot$
- $1 M_\odot$
- $0.6 M_\odot$
- $0.4 M_\odot$
- $0.8 M_\odot$
Luminosities range over 9 orders of magnitude (even though masses range by factor of \(~300\)). Luminosity is energy per time, and is a measure of how fast a star can burn off its mass. Because \(E \sim M\), we have \(t = \frac{E}{L} \sim M^{-3}\).

Massive stars consume their hydrogen much faster than lower mass stars.

*Main-sequence lifetimes decrease with increasing luminosity.*