This week we learn about AC circuits.

**Alternating Current**

So far, when learning about circuits we’ve treated the emf source as one of direct current, meaning that it always delivers a constant potential difference. In real world applications, many circuits are alternating current, or AC, with the potential difference (and thus the current varying as a function of time). The reason for using AC is primarily because it allows voltage levels to be stepped up/down using a transformer (recall that a transformer requires a *time dependent* current to work—this is what we get when dealing with AC).

Generically, we can refer to the device supplying potential in an AC circuit as an AC source. In circuit diagrams, these are drawn by the symbol in Figure 1. These supply a sinusoidally varying potential given by

\[ v = V \cos \omega t, \]  

(1)

where

• \( v \) is the *instantaneous* potential difference;
• \( V \) is the *maximum* potential difference; and
• \( \omega \) is the *angular* frequency.

Note that the angular frequency \( \omega \) (radians/second) is related to the “regular” frequency \( f \) (Hertz, or Hz) by

\[ \omega = 2\pi f. \]  

(2)

An example plot of AC voltage is shown in Figure 2.

Here in North America, commercial power systems use a frequency \( f = 60 \text{ Hz} \), (angular frequency \( \omega = 377 \text{ rad/s} \)). In Europe and much of the rest of the world, it’s \( f = 50 \text{ Hz} \) (\( \omega = 314 \text{ rad/sec} \)).

Similar to the voltage, the current in AC circuits varies sinusoidally,

\[ i = I \cos \omega t, \]  

(3)

where \( i \) and \( I \) are the instantaneous and maximum current, respectively.
**Phasors**

To represent alternating current (or voltage) graphically, we can use rotating vectors called phasors, which are drawn in phase diagrams. These are useful tools for visualizing current or voltage vs. time. In particular, it’s useful when we need to add or subtract alternating currents or voltages.

An example phase diagram is shown in Figure 3. Basically, what we do it to have a fixed-length vector whose total length is the maximum current $I$ (or maximum voltage $V$). This vector is constantly rotating around the $x$-$y$ axes, with the angle between the vector and the $x$ axis being $\omega t$. Then the $x$-axis projection of the vector is always equal to the instantaneous current (or voltage).

So for example, at $t = 0$, the vector lies completely along the $x$-axis; the projection is simply equal to its total length $I$. At $t = \omega/2\pi$, all of the vector lies along the $y$-axis, so the $x$-axis projection is zero and $i = 0$. Everywhere in between, the projection is some value between these two extremes.

**RMS**

It’s common to refer to AC voltages or currents in terms of their root-mean-square, or RMS, values. As demonstrated in Figure 4, these are given by

\[
V_{\text{rms}} = \frac{V}{\sqrt{2}} \quad (4)
\]

\[
I_{\text{rms}} = \frac{I}{\sqrt{2}}. \quad (5)
\]

**Meaning of the rms value** of a sinusoidal quantity (here, ac current with $I = 3$ A).

*We:
  1. graph current $i$ versus time,
  2. square the instantaneous current $i$,
  3. take the average (mean) value of $i^2$,
  4. take the square root of that average.*

Figure 3: A phase diagram.

Figure 4: Meaning of RMS currents & potentials.
AC circuits

Here we discuss what happens when you construct simple circuits with AC. First let’s discuss what happens when a resistor, an inductor, and a capacitor are connected to an AC source.

Resistor

Let’s imagine a resistor is connected to an AC source delivering current

\[ i = I \cos \omega t, \quad (6) \]

as in Figure 5(a). The potential drop across the resistor is then

\[ v = IR \cos \omega t, \quad (7) \]

i.e. a sinusoidal dependence with the same frequency as the current and maximum voltage \( V_R = IR \).

As shown in the Figure 5(b) the voltage and current have the same time dependence, although their amplitudes are different. To describe this, we say that they are “in phase”. It follows that their phasors are parallel to each other, just having different magnitudes, as in Figure 5(c).

Inductor

Now let’s imagine an inductor is connected to an AC source (Figure 6). The changing current will generate an emf according to

\[ \mathcal{E} = -L \frac{\Delta i}{\Delta t}. \]

This means that the potential difference across the inductor is given by

\[ v_L = L \frac{\Delta i}{\Delta t}, \quad (8) \]

What this means is that the potential drop across the inductor is equal to \( L \) times the rate of change of the current with respect to time.

The equation describing \( v_L \) can be solved using calculus to give

\[ v_L = -I \omega L \sin \omega t, \quad (9) \]

which is mathematically equivalent to

\[ v_L = I \omega L \cos (\omega t + 90^\circ). \quad (10) \]

As shown in Figure 6(b) and (c), the peaks in current and voltage are separated by \( \pi/2 \), or \( 90^\circ \). This is referred to as the phase. The phase is the angle describing the separation in peaks between voltage and current. Typically we refer to the phase of the voltage relative to the current. Here the voltage “leads”, or is ahead of the current by a phase of \( \pi/2 \), or \( 90^\circ \).

Important points

- The phase describes the time relationship between voltage and current.
- Reactance represents the voltage to current ratio through an inductor or a capacitor.

Important equations

- Resistor
  \[ i = I \cos \omega t \]
  \[ v = IR \cos \omega t \]

- Inductor
  \[ i = I \cos \omega t \]
  \[ v = I \omega L \cos (\omega t + 90^\circ) \]
  \[ X_L = \omega L \]

- Capacitor
  \[ i = I \cos \omega t \]
  \[ v = \left(\frac{I}{\omega C}\right) \cos (\omega t - 90^\circ) \]
  \[ X_C = \frac{1}{\omega C} \]
Figure 6: Inductor in an AC circuit.

More generally, we can talk about the phase angle $\phi$ and write the voltage as

$$v = V \cos (\omega t + \phi). \quad (11)$$

In the case of a single inductor, the phase angle is $\phi = 90^\circ$, and the amplitude is

$$V_L = I_0 \omega L. \quad (12)$$

With inductors, we can define a term called the reactance, $X_L$, which describes the ratio of the voltage amplitude to current amplitude. This is given by

$$X_L = \omega L. \quad (13)$$

Keep in mind that this is the ratio of amplitudes, or maximum values, not the instantaneous $v/i$ ratio. As a voltage/current ratio, the units of reactance are the same as resistance: the ohm ($\Omega$).

 Capacitor

Finally, let's consider a capacitor connected to an AC source (Figure 7). Since we have varying potential, we also have varying charge on the plates given by

$$\frac{\Delta q}{\Delta t} = \frac{1}{C} \Delta v = i. \quad (14)$$

Thus the rate of change of the potential is

$$\frac{\Delta v_C}{\Delta t} = \frac{1}{C} i \cos \omega t. \quad (15)$$

This can again be solved using calculus to give

$$v_C = \frac{i}{\omega C} \sin \omega t, \quad (16)$$

which is equivalent to

$$v_C = \frac{i}{\omega C} \cos (\omega t - 90^\circ). \quad (17)$$

Similar to the inductor case, the voltage and current are out of phase by 90°, except now the voltage is behind the current rather than ahead.
Hence we say that the voltage lags the current by $\pi/2$, or 90°. This is demonstrated in Figure 7(b) and (c). As suggested by Eq. (17), the voltage amplitude is given by

$$V_C = \frac{I}{\omega C}. \quad (18)$$

As with the inductor, we can define a reactance for the capacitor given by

$$X_C = \frac{1}{\omega C}. \quad (19)$$

**RLC Circuits**

Now let’s consider an RLC circuit containing a resistor, inductor, and capacitor all in series, as Figure 8. In this case Kirchhoff’s law still applies, and the algebraic sum of instantaneous potential drops equals the instantaneous source potential, i.e.

$$v = v_R + v_L + v_C. \quad (20)$$

More generally, the vector sum of the phasors $V_R$, $V_L$, and $V_C$ equals the phasor that describes the total voltage, $V$. This is shown in Figure 8(b) and (c). To form this vector sum, first subtract $V_C$ from $V_L$ since they point in opposite directions to each other. Then we’re just left with the resulting vector $(V_L - V_C)$ and $V_R$. These are always at a 90° angle, so we can get the magnitude of the sum using the Pythagorean theorem:

$$V = \sqrt{V_R^2 + (V_L - V_C)^2} \quad (21)$$

$$= \sqrt{(IR)^2 + (IX_L - IX_C)^2} \quad (22)$$

$$= I \sqrt{R^2 + (X_L - X_C)^2}. \quad (23)$$

The quantity $X_L - X_C$ is called the *reactance* of the circuit,

$$X = X_L - X_C. \quad (24)$$

We can also define a quantity called the *impedence* $Z$,

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad (25)$$

$$= \sqrt{R^2 + [\omega L - (1/\omega C)]^2}. \quad (26)$$

This has the property that

$$V = Iz. \quad (27)$$
Finally, each RLC circuit has a phase angle describing the voltage and current time-relationship. This is given by

\[
\tan \phi = \frac{V_L - V_C}{V_R}
\]

(28)

\[
= \frac{I(X_L - X_C)}{IR}
\]

(29)

\[
= \frac{X_L - X_C}{R}
\]

(30)

\[
= \frac{X}{R}
\]

(31)

\[
\Rightarrow \phi = \arctan \left( \frac{\omega L - 1/\omega C}{R} \right)
\]

(32)

Note that if \(X_L > X_C\), the voltage leads the current by an angle \(\phi\) between 0 and 90°. Conversely, if \(X_L < X_C\), the voltage lags the current.

Note that all of the relationships between voltage and current amplitudes (e.g. Eq. (27) and the like) are equally valid for RMS values, e.g.

\[
V_{\text{rms}} = I_{\text{rms}} Z.
\]

(33)

**Power in RLC circuits**

To understand the power delivered to RLC circuits, let’s first consider the simplest case of just a resistor \(R\) and a voltage source. In this case, the average power is equal to one-half of its maximum value, or

\[
P = \frac{1}{2} VI.
\]

(34)
Equivalently, we can write this in terms of RMS values,

\[ P = \frac{V}{\sqrt{2}} \cdot \frac{I}{\sqrt{2}} \]  
\[ = V_{\text{rms}} I_{\text{rms}}. \]  
\[ (35) \]

And since \( V_{\text{rms}} = R I_{\text{rms}} \), we can also write

\[ P = I_{\text{rms}}^2 R \]  
\[ = V_{\text{rms}}^2 / R \]  
\[ = V_{\text{rms}} I_{\text{rms}}. \]  
\[ (36) \]

Remember: these equations are only valid when there’s just a resistor in the circuit. **They cannot be used if an inductor or a capacitor is present.**

So what if an inductor and/or capacitor is present? Let’s consider the case with both an inductor and capacitor (and a resistor) hooked up to the voltage source. As with any AC circuit, the instantaneous power is given by

\[ p = vi \]  
\[ = [V \cos (\omega t + \phi)] [I \cos \omega t]. \]  
\[ (40) \]

From this, we can derive the expression for the average power,

\[ P = \frac{1}{2} VI \cos \phi \]  
\[ = V_{\text{rms}} I_{\text{rms}} \cos \phi. \]  
\[ (42) \]

In other words, the average power depends on the phase angle \( \phi \). We often call the quantity \( \cos \phi \) the power factor of the circuit. For a pure resistance circuit, \( \phi = 0 \), \( \cos \phi = 1 \), and \( P = V_{\text{rms}} I_{\text{rms}} \). For a pure resistanceless (\( L \) or \( C \) only) circuit, \( \phi = 90^\circ \), \( \cos \phi = 0 \), and \( P = 0 \). For a series RLC circuit, the power factor is equal to \( R / Z \).

From Eq. (43) and \( V_{\text{rms}} = I_{\text{rms}} Z \), it’s possible to derive equations similar to those we’ve seen before:

\[ P = V_{\text{rms}} I_{\text{rms}} \cos \phi \]  
\[ = I_{\text{rms}}^2 Z \cos \phi \]  
\[ = \frac{V_{\text{rms}}^2}{Z} \cos \phi. \]  
\[ (46) \]

**Series Resonance**

As demonstrated by Eq. (26), the impedance of an RLC circuit depends on its frequency:

\[ Z = \sqrt{R^2 + [\omega L - 1/(\omega C)^2]}. \]  
\[ (47) \]
Hence when the frequency is such that

\[ \omega L = 1 / (\omega C), \quad (48) \]

the impedance is equal to its smallest possible value, which is just the resistance \( R \). This is called the \textit{resonance angular frequency}, or \( \omega_0 \). It corresponds to the \textit{maximum possible current amplitude} for the circuit. To find the resonance frequency, just solve Eq. (48),

\[ \omega_0^2 = \frac{1}{LC}, \quad (49) \]

\[ \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}. \quad (50) \]

The \textit{resonance frequency}, \( f_0 \) is then \( \omega_0 / (2\pi) \).

When a circuit is “on resonance”, a few things are true:

- The currents are the \textit{same} in \( L \) and \( C \).
- The voltage in the inductor \textit{leads} the current by 90°.
- The voltage in the capacitor \textit{lags} the current by 90°.
- Hence the voltages across \( L \) and \( C \) always vary by 180°, or a half cycle.
  - If the amplitudes of the two voltages are equal, then they always sum to zero. The total voltage across the \( LC \) combination is zero.

Resonance in circuits is very similar to resonance in other systems in nature. As an example that you saw last semester, consider the forced oscillation of a harmonic oscillator. Here the amplitude of the \textit{mechanical} oscillation peaks when the driving frequency is close to the natural frequency of the system. This is basically the same thing that’s going on in circuits, except the natural frequency is now defined by the values of \( L \) and \( C \).
Example Problems
What is $I$?

$V_c = X_c I$

$\Rightarrow I = \frac{V_c}{X_c} = \frac{720\, \text{v}}{600\, \Omega} = 1.2\, \text{A}$

What is $Z$?

$Z = \frac{V}{I} = \frac{240\, \text{v}}{1.2\, \text{A}} = 200\, \Omega$

What is $X_L$ (2 values)?

$Z = \sqrt{R^2 + (X_L - X_C)^2}$

$\Rightarrow (X_L - X_C)^2 = Z^2 - R^2$
\[ X_L = X_C \pm \sqrt{Z^2 - R^2} \]

\[ X_L = 732.2 \ \Omega \]

or

\[ X_L = 467.7 \ \Omega \]
\[ \tan \phi = \frac{X_L - X_C}{\frac{Z}{i}} \]

\[ \phi = \arctan \left( \frac{500 \Omega - 300 \Omega}{300 \Omega} \right) = \arctan \left( \frac{2}{1} \right) = 63.4^\circ \]

Power factor:

\[ \cos \phi = \cos (63.4^\circ) = 0.832 \]

\[ P = V_{RMS} I_{RMS} \cos \phi \]

\[ = V_{RMS} \left( \frac{V_{RMS}}{Z} \right) \cos \phi \]

\[ \Rightarrow V_{RMS} = \sqrt{\frac{Z \cdot P}{\cos \phi}} \]

\[ Z = \sqrt{(300 \Omega)^2 + (500 \Omega - 300 \Omega)^2} = 360.6 \ \Omega \]

\[ \Rightarrow V_{RMS} = 161.2 \ \text{V} \]