Important points

• Changing magnetic flux induces a current in a closed circuit.
• The direction of the induced current flow is such that it opposes the change in magnetic flux (Lenz’s law).

Important equations

• Magnetic Flux (uniform field)
  \[ \Phi_B = B \cos \phi A \]
• Faraday’s law
  \[ \mathcal{E} = N |\Delta \Phi_B / \Delta t| \]
• Motional emf
  \[ \mathcal{E} = vBL \]

This week we learn about electromagnetic induction.

Magnetic Induction

This section deals with magnetic induction, the phenomenon by which changing magnetic fields can induce currents in circuits. This means that changing magnetic fields are an additional source of electromotive force, or emf.

First off, let’s introduce the concept of magnetic flux, which is entirely analogous to the electric flux we already saw.

Magnetic Flux

As illustrated in Figure 1, magnetic flux, \( \Phi_B \) is the amount of magnetic field passing through the perpendicular component of a surface with area \( A \).\(^1\) Mathematically, this means

\[ \Delta \Phi_B = B \cos \phi \Delta A. \] (1)

For a uniform field, this becomes

\[ \Phi_B = B \cos \phi A, \] (2)

or defining \( B_\perp \equiv B \cos \phi \),

\[ \Phi_B = B_\perp A. \] (3)

Note that for a surface parallel to the field, \( \phi = 90^\circ \) and \( \Phi_B = 0 \).

Faraday’s Law

As mentioned, changing magnetic fields can induce a current, or an emf, in a circuit. It turns out that the induced emf \( \mathcal{E} \) is directly proportional to the change in magnetic flux per unit time. This is called Faraday’s Law. Mathematically, Faraday’s law reads:

\[ \mathcal{E} = \left| \frac{\Delta \Phi_B}{\Delta t} \right|, \] (4)

where \( \Phi_B \) is the flux through the area of the circuit.

PHYS 202 Notes, Week 6
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February 23 & 25, 2016
Last updated: 02/25/2016 at 12:36:40

Figure 1: Magnetic flux: \( \Phi_B = BA \cos \phi \).

\(^1\) Magnetic flux has its own special SI unit the Weber, or Wb. 1Wb \( \equiv \) 1T·m\(^2\).
Equation (4) is the expression for the emf induced in a single closed loop. For a coil with \( N \) identical terms, it becomes

\[
\mathcal{E} = N \left| \frac{\Delta \Phi_B}{\Delta t} \right|. \tag{5}
\]

The change in magnetic flux per unit time can occur for different reasons. For example, you could have a stationary current loop and some magnetic field whose strength varies with time. Alternatively, you could have a constant magnetic field in some region of space and a loop being constantly moved into the field region. In this case, the area of the loop is changing, leading to an overall change in flux.

**Lenz’s Law**

Faraday’s law only deals with the magnitude of the emf (and in turn, current) generated by a changing \( \Phi_B \). To figure out the direction of the induced current, we turn to Lenz’s Law, which states:

*The current or emf induced by a changing magnetic flux is always in the direction that opposes that change in flux.*

What does this mean exactly? Recall that moving charges (currents) are the source of magnetic fields. So the induced current in some loop will itself create a magnetic field. The direction of the current will always be such that the newly created field opposes, or counteracts, the changing flux creating it. Figure 2 illustrates this for a few different example cases.
Motional emf

A related concept to Faraday/Lenz’s law is motional emf. This is illustrated in Figure 3, explained as follows:

- Consider a conducting metal rod moving in a uniform magnetic field, with a velocity $\vec{v}$ that’s perpendicular to the field ($\vec{v}$ to the right, $\vec{B}$ into the page).
- The magnetic field exerts an upwards force on the free positive charges in the conductor.
- The magnetic force causes positive charge to collect at the top of the rod and negative charge to collect at the bottom.
- The charge differences create an electric field which exerts a downward force $\vec{F}_e$ on the charges.
- The magnetic and electric forces balance, meaning that
  \[ F_e = F_B \]  
  \[ qE = qvB \]  
  \[ E = vB. \]  

- The potential difference between point $a$ at the top of the rod and point $b$ at the bottom of the rod is equal to $V_{ab} = EL$. Thus
  \[ V_{ab} = EL \]  
  \[ = vBL. \]  

Now what would happen if the rod was attached to a circuit loop as in Example 21.4? The potential difference $V_{ab}$ would act as an emf for the circuit with magnitude
  \[ \mathcal{E} = vBL, \]  
just as was figured out from Faraday’s law in the example. The direction of the emf will cause current to flow from point $a$ to point $b$. This corresponds to counter-clockwise in the loop of Example 21.4, a result that’s entirely consistent with Lenz’s law.

Mutual/Self Inductance and Transformers

**Mutual Inductance**

When two coils are placed together, a changing current in one of the coils will induce an emf in the other one. This coupling phenomenon between coils is referred to as mutual inductance.

An illustration of mutual inductance is shown in Figure 4. We have two coils, coil 1 and coil 2. Coil 1 has $N_1$ turns and coil 2 has $N_2$.

**Important equations**

- Mutual Inductance
  \[ M = \left| N_2 \Phi_{2i} / i_1 \right| \]  
  \[ = \left| N_1 \Phi_{1i} / i_2 \right| \]  
  \[ \mathcal{E}_1 = M \left| N_2 / i_1 \right| \]  
  \[ \mathcal{E}_2 = M \left| N_1 / i_2 \right| \]  
- Self Inductance
  \[ \mathcal{E} = L \left| N / M \right| \]  
- Transformers
  \[ v_2 / v_1 = N_2 / N_1 \]  
- Energy stored in an Inductor
  \[ U = LI^2 / 2 \]
Coil 1 carries some current $i_1$ that is varying with time. This current produces a magnetic field which passes through the area of coil 2, creating a flux $\Phi_{B2}$. As $i_1$ changes (with time), so does $B_1$, and in turn $\Phi_{B2}$. This generates an emf in coil 2 given by

$$E_2 = N_2 \frac{\Delta \Phi_{B2}}{\Delta t}. \quad (12)$$

Similarly, we could have a situation where coil 2 carries some current $i_2$ and creates a flux $\Phi_{B1}$ in coil 1. By symmetry with the first case, we get

$$E_1 = N_1 \frac{\Delta \Phi_{B1}}{\Delta t}. \quad (13)$$

It turns out that we can define a quantity called the mutual inductance which relates the two cases (either a current in coil 1 inducing an emf in coil 2 or vice-versa). The mutual inductance\(^2\) is given by

$$M = \left| \frac{N_2 \Phi_{B2}}{i_1} \right| = \left| \frac{N_1 \Phi_{B1}}{i_2} \right|. \quad (14)$$

From this, we can re-write Eqs. (12) & (13) as

$$E_2 = M \left| \frac{\Delta i_1}{\Delta t} \right| \quad (15)$$

and

$$E_1 = M \left| \frac{\Delta i_2}{\Delta t} \right|. \quad (16)$$

Transformers

Mutual inductance can be taken advantage of to build a device called a transformer, which can change the voltage levels in alternating-current (AC) circuits (i.e. circuits where the current is constantly changing). They are incredibly useful in real life, for example taking the $\sim 100,000$ volts used in power transmission lines and turning them into the 120 volts\(^3\) delivered in your household plugs.

\(^2\)The SI units of inductance are the Wb/A, or henry, H.

\(^3\)Or 220V in much of the world.
A transformer is made up of two coils as discussed in the last section on mutual inductance, with the same flux, \( \Phi_B \), in each coil. The induced emfs are then defined by

\[
\mathcal{E}_1 = N_1 \left| \frac{\Delta \Phi_B}{\Delta t} \right|, \\
\mathcal{E}_2 = N_2 \left| \frac{\Delta \Phi_B}{\Delta t} \right|.
\]

Thus \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are related by

\[
\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}.
\]

More generally, we can talk about the voltage levels “transformed” by a transformer:

\[
\frac{V_2}{V_1} = \frac{N_2}{N_1}.
\]

**Self Inductance**

When only a single coil is involved, we get a phenomenon called *self inductance*. This occurs because the field produced by a single coil will pass through its own area, creating a magnetic flux, and, in turn a *self induced emf*. This is illustrated in Figure 5.

In this case, we can define a term called the *self inductance*, or \( L \), according to the following equation:

\[
N |\Phi_B| = L |i|.
\]

If \( \Phi_B \) and \( i \) change with time, then

\[
N \left| \frac{\Delta \Phi_B}{\Delta t} \right| = L \left| \frac{\Delta i}{\Delta t} \right|.
\]

And combining Eqs. (20) & (21), we get the equation for the self-induced emf:

\[
\mathcal{E} = L \left| \frac{\Delta i}{\Delta t} \right|.
\]

The direction of the induced emf is found from Lenz’s law. Remember that the cause of the induced emf is *changing current* in the circuit. Hence the direction of the induced emf always serves to oppose that change.

The inductance of a coil is a constant that depends on its geometrical properties: size, shape, and number of turns. In circuits, we can design a specific element to have a particular inductance. This circuit element is called an *inductor*. We’ll learn more about these in a bit.
**Magnetic Field Energy**

Similar to capacitors, inductors can store energy. Except now the energy is stored in the form of magnetic fields rather than electric fields. The potential energy stored in an inductor with inductance $L$ carrying current $I$ is given by

$$U = \frac{1}{2} LI^2. \quad (23)$$

Doing some math, we can work out that the energy density, $u$, (potential energy per unit volume) depends on the magnetic field by

$$u = \frac{1}{2} \frac{B^2}{\mu_0}. \quad (24)$$

This is entirely analogous to the energy density in a capacitor:

$$u_{\text{capacitor}} = \frac{1}{2} \epsilon_0 E^2. \quad (25)$$

**Inductors in Circuits**

**RL Circuits**

Inductors are typically used as circuit devices. Figure 6 shows an example of an RL circuit, that is, a circuit containing a resistor, an inductor, and a constant emf source (like a battery).

When we close the switch in the circuit in Figure 6, the current varies with time according to:

$$i = \frac{\mathcal{E}}{R} \left[ 1 - e^{-(R/L)t} \right]. \quad (26)$$

Why is this? The reason comes from Kirchhoff’s loop rule, which states that the sum of potential differences along the loop must equal zero:

$$\mathcal{E} - iR - v_L = 0, \quad (27)$$

where $v_L$ is the potential drop across the inductor (from $b$ to $c$ in the drawing). From Eq. (22), this is equal to

$$v_L = L \frac{\Delta i}{\Delta t}. \quad (28)$$

Thus,

$$\mathcal{E} - iR - L \frac{\Delta i}{\Delta t} = 0. \quad (29)$$

Using the techniques of differential equations, Eq. (26) can be derived from Eq. (29). However, this is outside the scope of this course.

Let’s take note of the “immediate” ($t = 0$) and “long time” ($t \to \infty$) behavior of LC circuits. At $t = 0$, Eq. (26) becomes

$$i = 0. \quad (30)$$
Initially, there is no current flowing through the circuit; the inductor effectively acts like an open circuit element. At long times, Eq. (26) becomes

$$i = \frac{E}{R}.$$  \hspace{1cm} (31)

The circuit looks like we only have a resistor connected to a battery. The inductor is behaving just like a “short,” or a piece of ideal conducting wire.

As with the RC circuits we saw a couple of chapters ago, we can define a time constant by

$$\tau = \frac{L}{R}.$$  \hspace{1cm} (32)

A plot of current vs. time for an LC circuit is shown in Figure 7.

**LC Circuits**

Another type of circuit involving inductors is called an LC circuit, consisting of a capacitor and an inductor. In this arrangement, the current and the charge on the capacitor oscillate. Figure 8 illustrates what's going on:

1. Begin with the capacitor fully charged to some value $Q_{\text{max}}$ and zero current flow. All of the energy in the circuit is stored in the electric field.
between the capacitor plates.

2. The capacitor begins to discharge: current flows from the positive to the negative terminals of the capacitor. Due to the induced emf in the inductor coils, the discharge can’t happen instantly. Instead the current slowly builds up to some maximum value \( I_{\text{max}} \). The circuit’s energy is shared between the capacitor electric field and the magnetic field in the inductor.

3. When the current reaches \( I_{\text{max}} \), the charge on the plates is zero. All of the energy is stored in the magnetic field of the inductor.

4. The capacitor now begins to charge in the opposite sense, with positive charge on the formerly-negative plate and vice-versa. Again, this charge doesn’t happen instantly but rather over time. The circuit’s energy is shared between the capacitor electric field and the magnetic field in the inductor.

5. Eventually the capacitor becomes fully charged again in the opposite sense, and \( I = 0 \). All of the energy in the circuit is stored in the electric field between the capacitor plates.

6. Now we repeat steps (1–5), except in the opposite direction, ad infinitum.

So basically what’s happening is that we’re oscillating between no current and maximum charge on the capacitor; and maximum current and no charge on the capacitor. In between, we have some non-maximal charge and some non-maximal current. Similarly, the energy in the circuit oscillates, between being 100\% electric field in the capacitor and 100\% magnetic field in the inductor.

As with anything periodic, we can define a frequency for the oscillation. This is related to \( L \) and \( C \) by

\[
\omega = \sqrt{\frac{1}{LC}}.
\]  

(33)
Example Problems
**Example 21.4** (p. 676)

\[ \Phi_B = BLx; \quad \Delta \Phi_B = \frac{BLv}{\Delta t} = BLV \]

\[ \Rightarrow BLV = E \quad \text{"Motional emf"} \]

\[ E = (0.6\tau) (0.1\text{m}) (2.5 \text{ m/s}) \]

\[ = 0.15\text{ V} \]
\[
\Phi_B = NBA \cos \phi ; \quad \phi = 0^\circ \Rightarrow \cos \phi = 1
\]
\[
\Phi_B = (6 \times 10^{-5} \text{T})(0.0012 \text{ m}^2)(200)
\]
\[
= 1.44 \times 10^{-5} \text{ Wb}
\]

(a) Flux Before Notation:

(b) Flux After: \( \phi = 90^\circ \Rightarrow \cos \phi = 0 \)

\( \Phi = 0 \)

(c) Avg. EMF:

\[
\mathcal{E} = \frac{\Delta \Phi_B}{\Delta t} = \frac{1.44 \times 10^{-5} \text{ Wb}}{0.04 \text{ s}}
\]

\[
\Rightarrow \mathcal{E} = 0.36 \text{ mV}
\]
\[ L = 1.41 \text{ m} \]
\[ B = 1.2 \text{ T} \]
\[ v = 2.5 \text{ m/s} \]

(a) \( \text{Force is down.} \)
\[ \Rightarrow E = 0 \]

(b) \( E = vBL \)
\[ = (2.5 \text{ m/s}) (1.2 \text{ T}) (1.41 \text{ m}) \]
\[ = 4.23 \text{ V} \]

R is at higher potential since \( F_B \) is to the right.

(c) \( E = vBL \cdot \sin(60^\circ) \)
\[ = (4.23 \text{ V})(0.866) \]
\[ = 3.66 \text{ V} \]

Force is diagonal down/right
\[ \Rightarrow R \text{ at higher potential.} \]
\[ N = 125 \\
L = 4.5 \text{ mH} \]

7. \[ N|\Phi_B| = L|i|i| \]
   \[ \Rightarrow \Phi_B = \frac{4.5 \times 10^{-3} \text{ H}}{N} = \frac{(4.5 \times 10^{-3} \text{ H})(11.5\AA)}{125} \]
   \[ \Phi_B = 4.14 \times 10^{-4} \text{ Wb} \]

6. \[ \epsilon = L \left| \frac{\Delta i}{\Delta t} \right| \]
   \[ \Rightarrow \frac{\Delta i}{\Delta t} = \frac{\epsilon}{L} = \frac{1.16 \sqrt{2}}{4.5 \times 10^{-3} \text{ H}} = 25 \, \text{A/s} \]
\[ i = \frac{\varepsilon}{R} \left( 1 - e^{-\frac{R}{L}t} \right) \]

\[ i(0) = \frac{\varepsilon}{R} (1 - 1) = 0 \text{ A} \]

\[ V_i = (0 \text{ A}) (15 \text{ \Omega}) = 0 \text{ V} \]

\[ V_1 + V_2 = 25 \text{ V} \Rightarrow V_2 = 25 \text{ V} \]

\[ i = \frac{\varepsilon}{R} (1 - 0) = \frac{\varepsilon}{R} = 1.67 \text{ A} \]

\[ V_i = (1.67 \text{ A}) (15 \text{ \Omega}) = 25 \text{ V} \]

\[ V_2 + V_i = 25 \text{ V} \Rightarrow V_2 = 0 \]

\[ \text{Neither} \]
What about \( t = 0.8 \) ms?

\[
i = \frac{\varepsilon}{R} \left( 1 - e^{-\left(\frac{15}{12\times10^{-3}}\right)\left(0.8 \times 10^{-3}\right)} \right)
\]

\[
= \frac{\varepsilon}{R} \left( 1 - \frac{1}{e} \right)
\]

\[
= \frac{\varepsilon}{R} \left( 0.632 \right)
\]

\[
= 1.05 \text{ A}
\]

\[
V_1 = (i)(15 \text{ \Omega}) = 15.8 \text{ V}
\]

\[
V_1 + V_2 = 25 \text{ V} \Rightarrow V_2 = 9.2 \text{ V}
\]
\[ Q_{\text{max}} = ? \]

Use energy conservation

\[ U = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} \frac{(175 \mu C)^2}{15 \mu F} = 1.02 \times 10^{-3} \text{ J} \]

\[ U = \frac{1}{2} LI^2 \]

\[ \Rightarrow I = \sqrt{\frac{2U}{L}} = \sqrt{\frac{2(1.02 \times 10^{-3} \text{ J})}{5 \text{ mH}}} = 0.64 \text{ A} \]

\[ I_{\text{max}} \text{ means } Q = 0 \]

\[ V = \frac{Q}{C} \Rightarrow V_{\text{max}} = \frac{175 \mu C}{15 \mu F} = 11.67 \text{ V} \]

\[ i = 0 \text{ for max } Q, V. \]
Energy conservation:

\[ U_{\text{max}}^{(L)} = U_{\text{max}}^{(c)} = 1.02 \text{ mJ} \]

\[ I_{\text{max}} = 0.64 \text{ A} \]