Review Classical Mechanics

Four formulations of classical Mech.

Newton's Second Law

1) Newtonian : $p_i = m\dot{x}_i$; $\dot{p}_i = F_i = -\frac{\partial V}{\partial x_i} = m\ddot{x}_i$

2) Lagrangian : cartesian $x_i \longrightarrow$ generalized coord q_i

$$L(q_i, \dot{q}_i) = \sum_i \frac{1}{2} m \dot{q}_i^2 - V(q_i)$$

gen. momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

action

$$S[q_i(t)] = \int_{\{q_{i0}\}}^{\{q_{if}\}} dt \, L(q, \dot{q}_i)$$

the trajectories $q_i(t)$ are determined by

$$\frac{\delta S}{\delta q_i} = 0 \Rightarrow \text{Euler-Lagrange equation}$$

For HW#2

$$L = \frac{1}{2} m \vec{v}^2 + \frac{q}{c} \vec{v} \cdot \vec{A}$$

$$\vec{v} = \frac{d\vec{r}}{dt}$$

Lorentz force

$$m \frac{d^2\vec{r}}{dt^2} = \frac{q}{c} \vec{v} \times \vec{B}$$

$$\vec{B} = \vec{\nabla} \times A$$

$$\frac{d}{dt}(p_i) = \frac{\partial}{\partial q_i} L$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \Rightarrow \frac{d}{dt}(m\dot{q}) = -\frac{\partial V}{\partial q_i}$$

same dynamical eqs.

3) Hamiltonian :

$\{q_i, p_i\}$ as fundamental variables

$p_i = \frac{\partial L}{\partial \dot{q}_i}$ are ind. dyn. variables

$\dot{q}_i = \frac{p_i}{m}$ given p_i, the evolution of q_i is determine.

Hamilton's eq.

$$\boxed{H = \sum p_i \dot{q}_i - L(q_i, \dot{q}_i)}$$

eliminated all $\dot{q}_i$ in terms p_i.

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \left(= \frac{p_i}{m}\right)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \left(= -\frac{\partial V}{\partial q_i}\right)$$

$$= \sum_i p_i \frac{p_i}{m} - \left(\sum_i \frac{1}{2m} p_i^2 - V(q_i)\right)$$

$$= \sum_i \frac{1}{2m} p_i^2 + V(q_i)$$

4) Poissonian :

$$\frac{\partial}{\partial t} W(q_i, p_i) = \sum_i \left(\frac{\partial W}{\partial q_i} \frac{\partial q_i}{\partial t} + \frac{\partial W}{\partial p_i} \frac{\partial p_i}{\partial t} \right)$$

$$= \sum_i \left(\frac{\partial W}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial W}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = \{W, H\}$$

Poisson bracket

$$= \sum_i \Big(\underbrace{\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i}}_{\hat{T}} + \underbrace{\left(-\frac{\partial H}{\partial q_i}\right) \frac{\partial}{\partial p_i}}_{\hat{V}} \Big) W$$

$$\frac{\partial}{\partial t} W = (\hat{T} + \hat{V}) W$$

$$\boxed{W(t) = e^{t(\hat{T}+\hat{V})} W(0)}$$

$$\{q_i, q_j\} = 0 \quad \{p_i, p_j\} = 0 \quad \{q_i, p_j\} = \delta_{ij}$$

The free-particle case.

$$|\psi(t)\rangle = e^{-i\frac{t}{\hbar}\frac{1}{2m}\hat{p}^2}|\psi(0)\rangle$$

The exact solution in p-space

$$\psi(\vec{p},t) = \langle\vec{p}|\psi(t)\rangle = e^{-i\frac{t}{\hbar}\frac{1}{2m}\vec{p}^2}\underbrace{\langle p|\psi(0)\rangle}_{\psi(\vec{p},0)}$$

In r-space

$$\langle r|\psi(t)\rangle = \int\frac{d^3p}{(2\pi\hbar)^3}\langle r|e^{-i\frac{t}{\hbar}\frac{\hat{p}^2}{2m}}|\vec{p}\rangle\langle\vec{p}|\psi(0)\rangle$$

$|\psi(0)\rangle = |\vec{r}_0\rangle$

$$= \int\frac{d^3p}{(2\pi\hbar)^3}\langle\vec{r}|\vec{p}\rangle e^{-i\frac{t}{\hbar}\frac{\vec{p}^2}{2m}}\langle p|\psi(0)\rangle$$

$$\langle\vec{r}|\vec{r}_0\rangle = \int\frac{d^3p}{(2\pi\hbar)^3}e^{i\frac{\vec{p}\cdot\vec{r}}{\hbar}}e^{-i\frac{t}{\hbar}\frac{p^2}{2m}}\underbrace{e^{-i\frac{\vec{p}\cdot\vec{r}_0}{\hbar}}}_{\langle p|\vec{r}_0\rangle}$$

$$\langle\vec{r},t|\vec{r}_0,0\rangle = \int\frac{d^3p}{(2\pi\hbar)^3}e^{i\frac{t}{\hbar}\left[\vec{p}\cdot\frac{(\vec{r}-\vec{r}_0)}{t}-\frac{p^2}{2m}\right]}$$

$\vec{p}\cdot\dot{\vec{r}} - H = L$

$$= \int\frac{d^3p}{(2\pi\hbar)^3}e^{-i\frac{t}{\hbar}\frac{1}{2m}\left[p^2-2m\frac{\vec{p}\cdot(\vec{r}-\vec{r}_0)}{t}+\left(\frac{m(\vec{r}-\vec{r}_0)}{t}\right)^2-\left[\frac{m(\vec{r}-\vec{r}_0)}{t}\right]^2\right]}$$

$$\int_{-\infty}^{\infty}dp\,e^{-\frac{1}{2\tau}p^2} = \sqrt{2\pi\tau}$$

$$= \int\frac{d^3p}{(2\pi\hbar)^3}e^{-i\frac{t}{\hbar}\frac{1}{2m}\left[p-\frac{m(\vec{r}-\vec{r}_0)}{t}\right]^2}e^{i\frac{\cancel{t}}{\hbar}\frac{1}{2\cancel{m}}\frac{m^{\cancel{2}}(\vec{r}-\vec{r}_0)^2}{t^{\cancel{2}}}}$$

classical trajectory

$\tau = \frac{m\hbar}{ti}$

$$= \frac{1}{(2\pi\hbar)^3}\left(\frac{2\pi m\hbar}{ti}\right)^{3/2}e^{\frac{i}{\hbar}\frac{m(\vec{r}-\vec{r}_0)^2}{2t}}$$

$$\langle\vec{r},t|\vec{r}_0,0\rangle = \boxed{\left(\frac{m}{2\pi i\hbar t}\right)^{3/2}e^{\frac{i}{\hbar}\frac{m(\vec{r}-\vec{r}_0)^2}{2t}} = K(\vec{r}t;\vec{r}_0)}$$

Recall that classically

$$L = \frac{1}{2} m \dot{q}_i^2 = \frac{1}{2} m \vec{v}^2$$

with action $\qquad$ $\vec{v}$ = const, free particle

$$S = \int_{t_0}^{t} L\, dt = \frac{1}{2} m \vec{v}^2 (t - t_0)$$

$t_0 = 0$

$$= \frac{1}{2} m \frac{(\vec{r} - \vec{r}_0)^2}{t^2}\, t = \frac{1}{2} m \frac{(\vec{r} - \vec{r}_0)^2}{t}$$

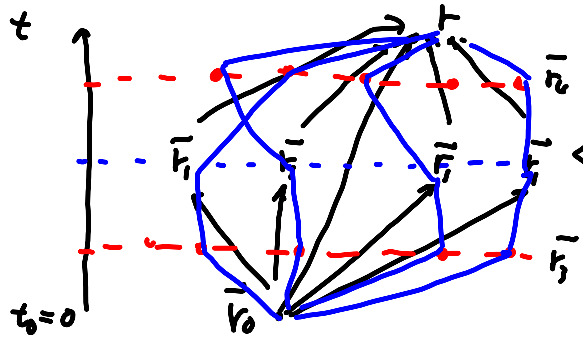
$$K(\vec{r}, t; \vec{r}_0 0) = \left(\frac{m}{2\pi i \hbar t}\right)^{3/2} e^{\frac{i}{\hbar} S(\vec{r}, t; \vec{r}_0 0)}$$

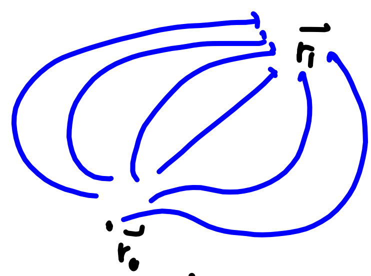
← classical action.

Quantum amplitude as a path integral (Feynman)

$$= \langle \vec{r} | e^{-i\frac{t}{\hbar}\hat{H}} | \vec{r}_0 \rangle$$

$$= \int d^3 r_1 \langle \vec{r} | e^{-i\frac{t/2}{\hbar}\hat{H}} | \vec{r}_1 \rangle \langle \vec{r}_1 | e^{-i\frac{t/2}{\hbar}\hat{H}} | \vec{r}_0 \rangle$$

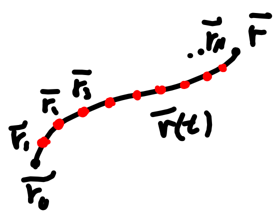




$$\langle \vec{r} t | \vec{r}_0, 0 \rangle = \sum_{\substack{\text{sum over} \\ \text{all paths } \vec{r}_0 - \vec{r}_0}} e^{\frac{i}{\hbar} S(\vec{r}, \vec{r}_0, t)}$$

$$\mathcal{D} = \prod_i \int d^3 \vec{r}_i \left(\sqrt{\frac{m}{2\pi i \hbar\, \delta t}}\right)^3$$

$$= \int_{\vec{r}_0}^{\vec{r}} \mathcal{D}(r(t))\, e^{\frac{i}{\hbar}\int_0^t L(\vec{r}, \dot{\vec{r}})\, dt}$$



With interaction $\qquad \epsilon = -i\frac{\Delta t}{\hbar}$

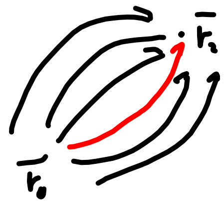
$$\underset{\text{in } \Delta t}{\langle \vec r_{n+1} | \vec r_n \rangle} = \langle \vec r_{n+1} | e^{\frac{1}{2}\epsilon \hat V} e^{\epsilon \hat T} e^{\frac{1}{2}\epsilon \hat V} | \vec r_n \rangle \qquad V(\hat{\vec r})$$

$$= \langle \vec r_{n+1} | e^{\epsilon \hat T} | \vec r_n \rangle \, e^{\frac{1}{2}\epsilon [V(\vec r_{n+1}) + V(\vec r_n)]} \qquad \hat T = \frac{\hat p^2}{2m}$$

$$= \left(\frac{m}{2\pi i \hbar \Delta t}\right)^{3/2} e^{\frac{i}{\hbar}\Delta t \left[\frac{1}{2}m\frac{(\vec r_{n+1} - \vec r_n)^2}{\Delta t^2} - V(\vec r_{n+1}, \vec r_n)\right]}$$

Path integral

$$\Rightarrow \boxed{\langle \vec r t, \vec r_0 0 \rangle = \int_{\vec r_0}^{\vec r} \mathcal{D}[\vec r(t)] \, e^{\frac{i}{\hbar}\int_0^t L(\vec r, \dot{\vec r}) dt}} \qquad \frac{1}{2}m\dot{\vec r}^2 - V = L$$



Since $\hbar \approx 0$ most important contribution is from paths which render the action stationary

$\Rightarrow \vec r(t)_{\text{classical}}$

1) Connection between classical trajectories & quantum amplitudes.

2) Alternative way of formulating QM without canonical quantization. $[\hat x_i, \hat p_j] = i\hbar$

Once we have $\langle \vec{r},t | \vec{r}_0,0 \rangle = K(\vec{r}t, \vec{r}_0 0)$

" propagator

$\langle \vec{r} | e^{-i\frac{t}{\hbar}\hat{H}} | \vec{r}_0 \rangle$

then

$$\langle r | \psi(t) \rangle = \langle r | e^{-i\frac{t}{\hbar}\hat{H}} | \psi(0) \rangle$$

$$\psi(\vec{r},t) = \int d^3\vec{r}_0 \, \langle r | e^{-i\frac{t}{\hbar}H} | \vec{r}_0 \rangle \langle \vec{r}_0 | \psi(0) \rangle$$

$$= \int d^3\vec{r}_0 \, K(\vec{r},t, \vec{r}_0 0) \, \psi(\vec{r}_0)$$

The path integral for a particle in a magnetic field

$$L = \frac{1}{2}mv^2 + \frac{q}{c}\vec{v}\cdot\vec{A} \qquad \vec{\nabla}\times\vec{A} = B$$

$$\langle \vec{r}t, \vec{r}_0 0\rangle = \int_{\vec{r}_0}^{\vec{r}} \mathcal{D}[\vec{r}(t)]\, e^{\frac{i}{\hbar}\int_0^t dt\left(\frac{1}{2}mv^2 + \frac{q}{c}\vec{A}\cdot\vec{v}\right)}$$

$$= \int_{\vec{r}_0}^{\vec{r}} \mathcal{D}[\vec{r}(t)]\, e^{\frac{i}{\hbar}\int_0^t dt\,\frac{1}{2}mv^2}\, e^{\frac{i}{\hbar}\int_{\vec{r}_0}^{\vec{r}} \cancel{dt}\,\frac{q}{c}\vec{A}\cdot\frac{d\vec{r}}{\cancel{dt}}}$$

$$= K(\vec{r}t, \vec{r}_0 0)\, e^{\frac{i}{\hbar}\frac{q}{c}\int_{\vec{r}_0}^{\vec{r}} \vec{A}\cdot d\vec{r}}$$

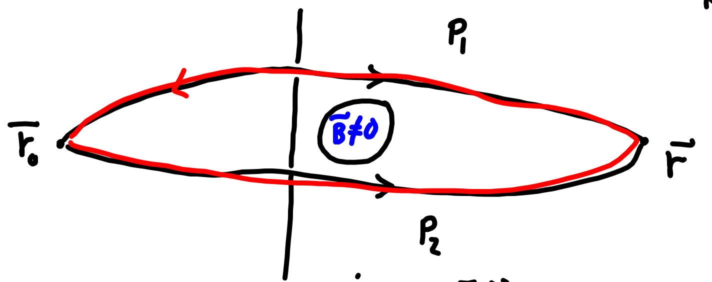
The Aharonov-Bohm effect:

Recall that

$$K_1 + K_2\, e^{\frac{i}{\hbar}\frac{q}{c}\oint \vec{A}\cdot d\vec{r}}$$

$$(K_1 + K_2 e^{i\phi})$$

$$(K_1^* + K_2^* e^{-i\phi})$$



$$\langle \vec{r}t|\vec{r}_0 0\rangle = K_{P_1}\, e^{\frac{i}{\hbar}\frac{q}{c}\int_{P_1}\vec{A}\cdot d\vec{r}} + K_{P_2}\, e^{\frac{i}{\hbar}\frac{q}{c}\int_{P_2}\vec{A}\cdot d\vec{r}}$$

Prob. $|\langle \vec{r}t|\vec{r}_0 0\rangle|^2 = |K_1|^2 + |K_2|^2 + K_1K_2^* e^{-i\phi} + K_2K_1^* e^{i\phi}$

$$= 1 + 1 + e^{i\varphi} + e^{-i\varphi}$$

$$= 2 + 2\cos\phi = 2(1+\cos\phi)$$

anyons

$$\varphi = \frac{q}{\hbar c}\oint \vec{A}\cdot d\vec{r} = \frac{q}{\hbar c}\Phi_m = \text{magnetic flux}$$

quantum effect $\propto \vec{A}$

Magnetic flux $\propto \oint \vec{A}\cdot d\vec{r}$ → causes interference even if the electron never enter into any region of magnetic field.

The uncertainty principle & the minimum uncertainty wave packet.

For any $|\psi\rangle$

$\langle \hat{x} \rangle = \langle \psi | \hat{x} | \psi \rangle$

$\langle \hat{x}^n \rangle = \langle \psi | \hat{x}^n | \psi \rangle$

$|\theta\rangle = (\hat{x} - \langle \hat{x} \rangle) |\psi\rangle$

$|\Phi\rangle = (\hat{p}_x - \langle \hat{p}_x \rangle) |\psi\rangle$

$\langle \theta | \theta \rangle = \langle \psi | (\hat{x} - \langle \hat{x} \rangle)^2 | \psi \rangle = \Delta x^2$

$\langle \Phi | \Phi \rangle = \langle \psi | (\hat{p}_x - \langle \hat{p}_x \rangle)^2 | \psi \rangle = \Delta p_x^2$

Δx = rms uncertainty in position

Δp_x = uncertainty in momentum

normalize $\rightarrow$ $|\alpha\rangle = \frac{1}{\Delta x} |\theta\rangle \quad |\beta\rangle = \frac{1}{\Delta p} |\Phi\rangle$

$$\left(\langle \alpha | - \lambda^* \langle \beta | \right) \left(|\alpha\rangle - \lambda |\beta\rangle \right) \geqslant 0$$

$\lambda = -i$

$$1 - \lambda^* \langle \beta | \alpha \rangle - \lambda \langle \alpha | \beta \rangle + |\lambda|^2 \geqslant 0$$

$$2 \geqslant \lambda \langle \alpha | \beta \rangle + \lambda^* \langle \beta | \alpha \rangle$$

$$2 \geqslant -i \left(\langle \alpha | \beta \rangle - \langle \beta | \alpha \rangle \right)$$

$$2 \geqslant -i \frac{1}{\Delta x \Delta p} \langle \underbrace{\hat{x}\hat{p}_x - \hat{p}_x \hat{x}}_{i\hbar} \rangle$$

$\langle \alpha | \beta \rangle$

$= \frac{1}{\Delta x} \frac{1}{\Delta p} \left((\hat{x} - \langle \hat{x} \rangle) (\hat{p}_x - \langle \hat{p}_x \rangle) \right)$

$= \frac{1}{\Delta x} \frac{1}{\Delta p} \left(\hat{x}\hat{p}_x - \langle \hat{x} \rangle \langle \hat{p}_x \rangle \right)$

$$2 \geqslant \frac{1}{\Delta x \Delta p} \hbar$$

or $\boxed{\Delta x \Delta p_x \geqslant \frac{1}{2} \hbar}$

Retrace our steps

$\hat{x} \rightarrow \hat{A} \quad \hat{p}_x \rightarrow \hat{B}$

$$\Delta A \Delta B \geqslant \frac{1}{2} (-i) \langle [\hat{A}, \hat{B}] \rangle$$

$$\Delta A \Delta B \geqslant \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|$$

no uncertainty if $[\hat{A}, \hat{B}] = 0$

The minimum uncertainty wave pack is

$$|\alpha\rangle - \lambda|\beta\rangle = 0 \qquad \lambda = -i$$

$$\frac{1}{\Delta x}\left(\hat{x} - \langle\hat{x}\rangle\right)|\psi\rangle + i\frac{1}{\Delta p}\left(\hat{p}_x - \langle p_x\rangle\right)|\psi\rangle = 0$$

$$\frac{1}{\Delta x}\left(x - \langle\hat{x}\rangle\right)\psi(x) + \cancel{i}\frac{1}{\Delta p}\left(\frac{\hbar}{\cancel{i}}\frac{\partial}{\partial x} - \langle p_x\rangle\right)\psi(x) = 0$$

$$\frac{1}{\Delta x}\left(x - \langle\hat{x}\rangle\right) + \frac{1}{\Delta p}\hbar\frac{\partial\psi}{\partial x}\frac{1}{\psi} - \frac{i}{\Delta p}\langle p_x\rangle = 0$$

$$\frac{\hbar}{\Delta p}\frac{\partial \ln\psi}{\partial x} = i\frac{1}{\Delta p}\langle p_x\rangle - \frac{1}{\Delta x}\left(x - \langle\hat{x}\rangle\right)$$

$\Delta p\,\Delta x = \frac{1}{2}\hbar$

$$\cancel{\frac{\hbar}{\Delta p}}\ln\psi = i\frac{1}{\cancel{\Delta p}_{\hbar}}\langle p_x\rangle x - \frac{\Delta p}{\hbar}\frac{1}{\Delta x}\left[\frac{x^2}{2} - \langle x\rangle x + c\right]$$

$\Delta p = \frac{1}{2}\frac{\hbar}{\Delta x}$

$$\boxed{\psi(x) = e^{i\frac{p_x x}{\hbar}}\, e^{-\frac{1}{4}\frac{1}{\Delta x^2}\left(x - \langle x\rangle\right)^2}}$$

min uncertainty w.f.