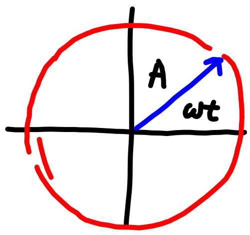
The harmonic oscillator

↪ algebraic method

↪ creation & annihilation operators:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x$$



Consider: $A = \sqrt{\frac{m}{2}}\,\omega x + i \frac{1}{\sqrt{2m}} p$

$$\dot{A} = \sqrt{\frac{m}{2}}\,\omega \frac{p}{m} + i \frac{1}{\sqrt{2m}} (-m\omega^2 x)$$

$$= \omega \left(\frac{p}{\sqrt{2m}} - i \sqrt{\frac{m}{2}}\,\omega x \right) = -i\omega A$$

$$A = e^{-i\omega t} A_0 \qquad |A| = \sqrt{E}$$

Quantum Mechanically

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

$$= \hbar\omega\left[-\frac{\hbar}{2m\omega}\frac{d^2}{dx^2} + \frac{1}{2}m\frac{\omega}{\hbar}x^2\right]$$

$\hat{H} = \hbar\omega$

$$\hbar\omega\left[\sqrt{\frac{m\omega}{2\hbar}}x - \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\right]\left[\sqrt{\frac{m\omega}{2\hbar}}x + \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\right]$$

$$\hbar\omega\left[\underbrace{\sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\sqrt{\frac{1}{2m\omega\hbar}}\hat{p}}_{a^+}\right]\left[\underbrace{\hat{x} + \hat{p}}_{a}\right] \qquad \frac{d}{dx} = i\frac{\hat{p}}{\hbar}$$

$$a^+a = \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{1}{2m\omega\hbar}\hat{p}^2 + (\underbrace{\hat{x}\hat{p}-\hat{p}\hat{x}}_{i\hbar})\frac{1}{2}\frac{i}{\hbar}$$

$$= \frac{\hat{H}}{\hbar\omega} \qquad -\frac{1}{2}$$

$$\boxed{\hat{H} = \hbar\omega\left(a^+a + \frac{1}{2}\right)}$$

a^+ = creation op

a = annihilation

Since $\hat{N} = a^+ a \propto \hat{H}$

define eigen state of $\hat{N}$

$$\hat{N}|n\rangle = n|n\rangle$$

with energy $E = (n+\frac{1}{2})\hbar\omega$

$\boxed{n \geq 0}$

$$\langle n|\hat{N}|n\rangle = n$$

$$\langle n|a^+ a|n\rangle = \langle\phi|\phi\rangle \geq 0$$

$= 0$ only if $|\phi\rangle = 0$

show now n must be a whole number.

$$[a, a^+] = [\alpha\hat{x} + i\beta\hat{p}, \alpha\hat{x} - i\beta\hat{p}]$$

$$= -i\alpha\beta[\hat{x},\hat{p}] + i\alpha\beta[\hat{p},\hat{x}] = -i\,2\underbrace{[\hat{x},\hat{p}]}_{i\hbar}\underbrace{\alpha\beta}_{\frac{1}{2\hbar}}$$

$$= 1$$

$$a a^+ - a^+ a = 1 \Rightarrow \boxed{a a^+ = \hat{N} + 1}$$

$$\hat{N} a = \underbrace{a^+ a}_{a a^+ - 1} a = [a a^+ - 1] a = a[a^+ a - 1] = a(\hat{N} - 1)$$

$$\hat{N}\underbrace{a|n\rangle} = a(\hat{N} - 1)|n\rangle$$

$$= (n-1)\underbrace{a|n\rangle}$$

The state $a|n\rangle$

has eigenvalue $(n-1)$ when $|n\rangle$ has eigenvalue n

Therefore if $\langle n|n\rangle = 1$

$$\langle n|\hat{N}|n\rangle = n$$

$$\langle n|a^+ a|n\rangle = n$$

$$|n-1\rangle = \frac{a|n\rangle}{\sqrt{n}}$$

$$\boxed{a|n\rangle = \sqrt{n}\,|n-1\rangle}$$

so that $\langle n-1|n-1\rangle = 1$

$$a^2|n\rangle = \sqrt{n}\,a|n-1\rangle = \sqrt{n}\sqrt{n-1}\,|n-2\rangle \text{ etc.}$$

$$a|1\rangle = |0\rangle \qquad a|0\rangle = 0$$

ground state with $E = \frac{1}{2}\hbar\omega$

If ξ is $0 < \xi < 1$

$$a|\xi\rangle = \sqrt{\xi}\,|\xi - 1\rangle$$

no such state

Similarly $a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$

All states $|n\rangle$ can now be constructed by

$$|1\rangle = \frac{a^{+}|0\rangle}{\sqrt{1}}$$

$$|2\rangle = \frac{a^{+}|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2!}}(a^{+})^{2}|0\rangle$$

$$\vdots$$

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(a^{+})^{n}|0\rangle}$$

Expectation values

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\langle n'|a|n\rangle = \sqrt{n}\,\langle n'|n-1\rangle = \sqrt{n}\,\delta_{n'\,n-1}$$

$$a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\langle n'|a^{+}|n\rangle = \sqrt{n+1}\,\delta_{n'\,n+1}$$

$$\langle n'|x|n\rangle = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\,\langle n'|\,a+a^{+}|n\rangle$$

$$= \sqrt{v}\,\left(\sqrt{n}\,\delta_{n'\,n-1} + \sqrt{n+1}\,\delta_{n'\,n+1}\right)$$

$$\langle n'|x^2|n\rangle = v\,\langle n'|(a+a^{+})^2|n\rangle$$

$$= v\,\langle n'|\; aa + \underbrace{aa^{+} + a^{+}a}_{2\hat{N}+1} + a^{+}a^{+}\,|n\rangle$$

$$= v\,(2n+1)\,\delta_{n'n}$$

$$+ v\left[\sqrt{n}\sqrt{n-1}\,\delta_{n'\,n-2} + \sqrt{n+2}\sqrt{n+1}\,\delta_{n',n+2}\right]$$