

## Relativistic motion in a constant electromagnetic field

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For a relativistic charged particle moving in a constant electromagnetic field, its velocity 4-vector has been well studied. However, despite the fact that both the electromagnetic field and the equations of motion are purely real, the resulting 4-velocity is seemingly due to a complex electromagnetic field. This work shows that this is not due to some complex formalism used (such as Clifford algebra) but is intrinsically due to the fact that the  $o(3,1)$  Lie algebra of the Lorentz group is equivalent to two commuting complex  $su(2)$  algebras. Expressing the complex  $su(2)$  generators in terms of the boost and rotation operators then naturally introduces a complex electromagnetic field. This work solves the equation of motion not as a matrix equation, but as an operator evolution equation in terms of the generators of the Lorentz group. The factorization of the real evolution operator into two commuting complex evolution operators then directly gives the time evolution of the velocity 4-vector without any reference to an intermediate field. © 2009 American Institute of Physics. [DOI: 10.1063/1.3064796]

### I. INTRODUCTION

The equations of motion of a negatively charged particle  $q=-e$  moving in a constant electromagnetic field  $F_{\mu\nu}$  are given by

$$\frac{du_{\mu}}{d\tau} = \frac{e}{mc} u_{\nu} F_{\nu\mu} \quad (1)$$

and

$$\frac{dx_{\mu}}{d\tau} = u_{\mu}, \quad (2)$$

where  $x_{\mu}=(x_0, \mathbf{r})$ ,  $x_0=ct$ ,  $u_{\mu}=(u_0, \mathbf{u})$ , and  $\tau$  is the proper time. Once (1) is known, the integration of (2) is straightforward. This is a well studied problem, with many published solutions.<sup>1-9</sup> Since (1) is just a matrix equation, it can be directly integrated, as done by Taub,<sup>1</sup> Hellwig,<sup>2</sup> and Hyman.<sup>3</sup> However, the algebraic manipulations used were purely formal and gave no insight on why the solution is the way it is. Bacry *et al.*<sup>4</sup> offered a group theoretic analysis of the problem, but the explicit form of the 4-velocity solution was not given. More recently, Hestenes,<sup>5</sup> Salingaros,<sup>6</sup> Baylis and Jones,<sup>7</sup> and Zeni and Rodrigues<sup>8,9</sup> applied Clifford algebra techniques to solve this problem. Salingaros gave a detailed comparison with earlier results of Taub and Hestenes, correcting some discrepancies. However, his explicit solution is only for a particle starting from rest. Baylis and Jones' solution for an arbitrary initial 4-velocity is very abbreviated. Zeni and Rodrigues' solution is similar in form to the solution presented here, but the derivations are completely different.

Salingaros' work highlighted the fact that, despite the equation of motion (1) and the electromagnetic field are both real, the resulting 4-velocity is seemingly due to a complex electromag-

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netic field. Since complex fields are a natural part of Clifford algebra,<sup>5-9</sup> it is unclear whether the solution merely reflects the formalism used or that a complex electromagnetic field is an intrinsic part of the solution.

In this work, we solve (1) by an entirely elementary method without invoking any advance formalism such as Clifford algebra. Instead of solving (1) as a matrix equation, we solve it via an evolution operator, as it is done in the Poisson bracket formulation of symplectic integrators.<sup>10-14</sup> Just like the quantum evolution operator, which acts on any quantum state and evolve it forward in time, the classical evolution operator acts on any dynamical variable and evolve it forward in time. The method of classical evolution operator is described in Sec. II. For a relativistic particle in a constant electromagnetic field, the corresponding evolution operator is the exponential of generators of the Lorentz group. Since the  $o(3,1)$  Lie algebra of the Lorentz group generators is equivalent to two commuting complex  $su(2)$  algebras, the evolution operator can be factored into two complex evolution operators. In each complex evolution operator, re-expressing the complex  $su(2)$  generators as boost and rotation operators then naturally introduces a complex electromagnetic field. Thus the complex electromagnetic field is an intrinsic part of the solution due to the structure of the Lorentz group and is independent of any Clifford algebra formalism. This is shown in Sec. III. The explicit velocity solution, for an arbitrary initial 4-velocity vector, is given in Sec. IV. Some conclusions and applications are indicated in Sec. V.

## II. CLASSICAL EVOLUTION OPERATOR

Equation (1) can be written out explicitly as

$$\frac{d\mathbf{u}}{d\tau} = \frac{e}{mc}(\mathbf{B} \times \mathbf{u} - \mathbf{E}u_0), \quad (3)$$

$$\frac{du_0}{d\tau} = \frac{e}{mc}(-\mathbf{E} \cdot \mathbf{u}). \quad (4)$$

The evolution of any dynamical variable  $W(x_\mu, u_\mu)$  (including  $x_\mu$  and  $u_\mu$  themselves) is given by

$$\frac{dW}{d\tau} = \frac{\partial W}{\partial x_\mu} \frac{dx_\mu}{d\tau} + \frac{\partial W}{\partial u_\mu} \frac{du_\mu}{d\tau}, \quad (5)$$

$$= \left( u_\mu \frac{\partial}{\partial x_\mu} + \frac{e}{mc} u_\nu F_{\nu\mu} \frac{\partial}{\partial u_\mu} \right) W = (T + V)W. \quad (6)$$

If  $dx_\mu/d\tau$  and  $du_\mu/d\tau$  are derivable from a Hamiltonian, then the right-hand side of (5) is just the Poisson bracket. However, as realized in Ref. 15, as long as one has the equations of motion, one can define an evolution operator regardless whether one has a canonical Hamiltonian. Equation (6) has the operator solution,

$$W(x_\mu(\tau), u_\mu(\tau)) = e^{\tau(T+V)} W(x_\mu, u_\mu). \quad (7)$$

In particular, because  $F_{\mu\nu}$  is constant,  $(T+V)u_\mu = V^n u_\mu$ , the 4-velocity is evolved simply by

$$u_\mu(\tau) = e^{\tau(T+V)} u_\mu = e^{\tau V} u_\mu. \quad (8)$$

From the explicit forms (3) and (4), one sees that

$$\tau V = \frac{e\tau}{mc} \left( (\mathbf{B} \times \mathbf{u}) \cdot \frac{\partial}{\partial \mathbf{u}} - \mathbf{E} \cdot \mathbf{u} \frac{\partial}{\partial u_0} - u_0 \mathbf{E} \cdot \frac{\partial}{\partial \mathbf{u}} \right) = \xi (\mathbf{B} \cdot \mathbf{J} + \mathbf{E} \cdot \mathbf{K}), \quad (9)$$

where  $\xi = e\tau/mc$  is the scaled proper time, and

$$\mathbf{J} = \mathbf{u} \times \frac{\partial}{\partial \mathbf{u}} \quad \mathbf{K} = -\mathbf{u} \frac{\partial}{\partial u_0} - u_0 \frac{\partial}{\partial \mathbf{u}} \quad (10)$$

are the three rotation and three boost generators of the Lorentz group. If  $\mathbf{E}=0$ , then

$$u_0(\xi) = \exp(\xi \mathbf{B} \cdot \mathbf{J}) u_0 = u_0, \quad (11)$$

$$\mathbf{u}(\xi) = \exp(\xi \mathbf{B} \cdot \mathbf{J}) \mathbf{u} = \exp\left(\xi B (\hat{\mathbf{B}} \times \mathbf{u}) \cdot \frac{\partial}{\partial \mathbf{u}}\right) \mathbf{u}, \quad (12)$$

where  $\hat{\mathbf{B}} = \mathbf{B}/B$  is the unit vector of the magnetic field. Using the above form (12), as shown in Ref. 16, the expansion of the exponential then directly gives the finite rotation

$$\mathbf{u}(\xi) = \mathbf{u} + \sin(\xi B) (\hat{\mathbf{B}} \times \mathbf{u}) + (1 - \cos(\xi B)) \hat{\mathbf{B}} \times (\hat{\mathbf{B}} \times \mathbf{u}). \quad (13)$$

Similarly, if  $\mathbf{B}=0$ , then the exponential operator gives

$$\begin{aligned} u_0(\xi) &= \exp(\xi \mathbf{E} \cdot \mathbf{K}) u_0 = \left(1 + \frac{1}{2}(\xi E)^2 + \dots\right) u_0 - \left(\xi E + \frac{1}{3!}(\xi E)^3 + \dots\right) \hat{\mathbf{E}} \cdot \mathbf{u} \\ &= \cosh(\xi E) u_0 - \sinh(\xi E) \hat{\mathbf{E}} \cdot \mathbf{u}, \end{aligned} \quad (14)$$

$$\mathbf{u}(\xi) = \exp(\xi \mathbf{E} \cdot \mathbf{K}) \mathbf{u} = \mathbf{u} + \hat{\mathbf{E}} ((\cosh(\xi E) - 1) \hat{\mathbf{E}} \cdot \mathbf{u} - \sinh(\xi E) u_0). \quad (15)$$

Equations (14) and (15) correspond to a general Lorentz boost in the direction of  $\hat{\mathbf{E}} = \mathbf{E}/E$  preserving the Minkowski norm  $u_0^2(\xi) - |\mathbf{u}(\xi)|^2 = u_0^2 - |\mathbf{u}|^2$ . The reason for the choice of  $q = -e$  is that the rotation is then right handed and the Lorentz boost is the standard transformation rather than its inverse.

If both  $\mathbf{B}$  and  $\mathbf{E}$  are nonvanishing, then the general solution for the velocity 4-vector is given by

$$u_\mu(\xi) = e^{\xi(\mathbf{B} \cdot \mathbf{J} + \mathbf{E} \cdot \mathbf{K})} u_\mu. \quad (16)$$

From the study of symplectic integrators,<sup>10-14</sup> such an exponential of two operators can be approximated to any order in  $\xi$  via the product decomposition,

$$e^{\xi(\mathbf{B} \cdot \mathbf{J} + \mathbf{E} \cdot \mathbf{K})} = \prod_i e^{b_i \xi \mathbf{B} \cdot \mathbf{J}} e^{e_i \xi \mathbf{E} \cdot \mathbf{K}}, \quad (17)$$

with suitable coefficients  $\{b_i, e_i\}$ . Since the effect of each exponential is known (11)–(15), the general product can be computed in sequence. The two elementary second order approximations are

$$e^{\xi(\mathbf{B} \cdot \mathbf{J} + \mathbf{E} \cdot \mathbf{K})} = e^{(\xi/2) \mathbf{B} \cdot \mathbf{J}} e^{\xi \mathbf{E} \cdot \mathbf{K}} e^{(\xi/2) \mathbf{B} \cdot \mathbf{J}} + O(\xi^3) = e^{(\xi/2) \mathbf{E} \cdot \mathbf{K}} e^{\xi \mathbf{B} \cdot \mathbf{J}} e^{(\xi/2) \mathbf{E} \cdot \mathbf{K}} + O(\xi^3). \quad (18)$$

This approach is purely real, no complex quantity enters anywhere. However, since we are interested only in the exact evaluation of (16), higher order approximations of the form (17) will not be considered here.

### III. COMPLEX DECOMPOSITION

The Lorentz group generators (10) obey the well known commutator relations

$$[J_i, J_j] = -\varepsilon_{ijk} J_k \quad [J_i, K_j] = -\varepsilon_{ijk} K_k \quad [K_i, K_j] = \varepsilon_{ijk} J_k. \quad (19)$$

This implies that

$$[\mathbf{B} \cdot \mathbf{J}, \mathbf{E} \cdot \mathbf{K}] = (-\mathbf{B} \times \mathbf{E}) \cdot \mathbf{K}. \quad (20)$$

Thus if  $\mathbf{B} \times \mathbf{E} = 0$ , then  $\mathbf{B} \cdot \mathbf{J}$  and  $\mathbf{E} \cdot \mathbf{K}$  commute and the solution is given by

$$u_\mu(\xi) = e^{\xi \mathbf{B} \cdot \mathbf{J}} e^{\xi \mathbf{E} \cdot \mathbf{K}} u_\mu. \quad (21)$$

This is the common starting point of many solutions.<sup>5,6</sup> The remaining task is then to transform to such a Lorentz frame in which  $\mathbf{B} \times \mathbf{E} = 0$ , apply the solution (21) and transform back.<sup>5,6</sup> However, there is a much more direct way of decomposing  $\mathbf{B} \cdot \mathbf{J} + \mathbf{E} \cdot \mathbf{K}$  into two commuting operators based on the fact that the Lie algebra of  $o(3,1)$  is equivalent to that of  $su(2) \times su(2)$ . The complex operators

$$\mathbf{M} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}) \quad \text{and} \quad \mathbf{M}^* = \frac{1}{2}(\mathbf{J} - i\mathbf{K}) \quad (22)$$

have commutators,

$$[M_i, M_j] = -\varepsilon_{ijk} M_k \quad [M_i^*, M_j^*] = -\varepsilon_{ijk} M_k^* \quad [M_i, M_j^*] = 0. \quad (23)$$

Thus we have an exact, but complex decomposition

$$e^{\xi(\mathbf{B} \cdot \mathbf{J} + \mathbf{E} \cdot \mathbf{K})} = e^{\xi(\mathbf{B} + i\mathbf{E}) \cdot \mathbf{M}^*} e^{\xi(\mathbf{B} - i\mathbf{E}) \cdot \mathbf{M}}. \quad (24)$$

In (24), one notices that

$$e^{\xi(\mathbf{B} - i\mathbf{E}) \cdot \mathbf{M}} = e^{\xi(\mathbf{B} - i\mathbf{E}) \cdot (\mathbf{J} + i\mathbf{K})/2} \quad (25)$$

corresponding to having a complex electric field  $\mathbf{F}/2 = (\mathbf{E} + i\mathbf{B})/2$  and a complex magnetic field  $-i\mathbf{F}/2 = (\mathbf{B} - i\mathbf{E})/2$ . Since  $-i\mathbf{F} \times \mathbf{F} = 0$ , one can further decompose

$$e^{\xi(\mathbf{B} - i\mathbf{E}) \cdot (\mathbf{J} + i\mathbf{K})/2} = e^{\xi(\mathbf{F}/2) \cdot \mathbf{K}} e^{\xi(-i\mathbf{F}/2) \cdot \mathbf{J}}. \quad (26)$$

Similarly,

$$e^{\xi(\mathbf{B} + i\mathbf{E}) \cdot (\mathbf{J} - i\mathbf{K})/2} = e^{\xi(-i\mathbf{F}/2)^* \cdot \mathbf{J}} e^{\xi(\mathbf{F}/2)^* \cdot \mathbf{K}}. \quad (27)$$

Thus the complex decomposition of the purely real evolution operator  $\exp[\xi(\mathbf{B} \cdot \mathbf{J} + \mathbf{E} \cdot \mathbf{K})]$  naturally introduces the complex electromagnetic field  $\mathbf{F} = \mathbf{E} + i\mathbf{B}$ . Furthermore, the evolution operator consists of four pieces, best evaluated in pair as in (26) and (27). Note that

$$e^{\xi(\mathbf{F}/2)^* \cdot \mathbf{K}} e^{\xi(\mathbf{F}/2) \cdot \mathbf{K}} \neq e^{\xi \mathbf{E} \cdot \mathbf{K}} \quad (28)$$

and the evaluation of (26) and (27) cannot be further simplified. The complex decomposition has been used by Fredsted<sup>16</sup> to obtain a spinor representation of the Lorentz group, but he did not solve the charged particle problem as it is done here.

#### IV. EXPLICIT SOLUTION

In order to evaluate (26) and (27) using our previous results (11)–(15), it is only necessary to define the norm of the complex vector  $\mathbf{F} = \mathbf{E} + i\mathbf{B}$  as

$$F = \sqrt{\mathbf{F} \cdot \mathbf{F}} = \sqrt{|\mathbf{E}|^2 - |\mathbf{B}|^2 + i2(\mathbf{E} \cdot \mathbf{B})} \quad (29)$$

and the corresponding complex unit vector as  $\hat{\mathbf{F}} = \mathbf{F}/F$ . For  $z = x + iy = |z|e^{i\theta}$ ,  $\sqrt{z} = u + iv$ , where

$$\begin{aligned}
 u &= |z|^{1/2} \cos(\theta/2) = \sqrt{|z|(1 + \cos \theta)/2} = \sqrt{(|z| + x)/2}, \\
 v &= |z|^{1/2} \sin(\theta/2) = \sqrt{|z|(1 - \cos \theta)/2} = \sqrt{(|z| - x)/2}.
 \end{aligned}
 \tag{30}$$

For the ease of later comparison, we will follow Salingaros' notation and define

$$\kappa_1 = |\mathbf{E}|^2 - |\mathbf{B}|^2, \quad \kappa_2 = 2(\mathbf{E} \cdot \mathbf{B}), \quad \kappa = \sqrt{\kappa_1^2 + \kappa_2^2}, \tag{31}$$

$$E' = \sqrt{\frac{\kappa + \kappa_1}{2}}, \quad \text{and} \quad B' = \sqrt{\frac{\kappa - \kappa_1}{2}} \tag{32}$$

Thus

$$F = E' + iB' \quad \text{and} \quad F^*F = (E')^2 + (B')^2 = \kappa. \tag{33}$$

Since two exponential operators in (26) commute, their effects simply add,

$$\begin{pmatrix} u'_0 \\ \mathbf{u}' \end{pmatrix} = e^{\xi(\mathbf{F}/2) \cdot \mathbf{K}} e^{\xi(-\mathbf{F}/2) \cdot \mathbf{J}} \begin{pmatrix} u_0 \\ \mathbf{u} \end{pmatrix} \tag{34}$$

resulting in

$$u'_0 = \cosh(\xi F/2) u_0 - \sinh(\xi F/2) \hat{\mathbf{F}} \cdot \mathbf{u}, \tag{35}$$

$$\begin{aligned}
 \mathbf{u}' &= \mathbf{u} + \hat{\mathbf{F}}((\cosh(\xi F/2) - 1) \hat{\mathbf{F}} \cdot \mathbf{u} - \sinh(\xi F/2) u_0) + \sin(-i\xi F/2) (\hat{\mathbf{F}} \times \mathbf{u}) \\
 &\quad + (1 - \cos(-i\xi F/2)) \hat{\mathbf{F}} \times (\hat{\mathbf{F}} \times \mathbf{u}).
 \end{aligned}$$

The latter now *greatly* simplifies to

$$\mathbf{u}' = \cosh(\xi F/2) \mathbf{u} - \sinh(\xi F/2) (u_0 \hat{\mathbf{F}} + i \hat{\mathbf{F}} \times \mathbf{u}). \tag{36}$$

One can check that the cross-product term above is essential for preserving the Minkowski norm  $(u'_0)^2 - (\mathbf{u}')^2 = u_0^2 - \mathbf{u}^2$ . To further simplify the notation, let us denote

$$c = \cosh(\xi F/2) \quad \text{and} \quad s = \sinh(\xi F/2), \tag{37}$$

then, finally,

$$\begin{pmatrix} u_0(\xi) \\ \mathbf{u}(\xi) \end{pmatrix} = e^{\xi(-i\mathbf{F}/2)^* \cdot \mathbf{J}} e^{\xi(\mathbf{F}/2)^* \cdot \mathbf{K}} \begin{pmatrix} u'_0 \\ \mathbf{u}' \end{pmatrix}, \tag{38}$$

where

$$u_0(\xi) = c^* u'_0 - s^* \hat{\mathbf{F}}^* \cdot \mathbf{u}' = (c^* c + s^* s \hat{\mathbf{F}}^* \cdot \hat{\mathbf{F}}) u_0 - (2 \operatorname{Re}(c^* s \hat{\mathbf{F}}) - i s^* s \hat{\mathbf{F}}^* \times \hat{\mathbf{F}}) \cdot \mathbf{u}, \tag{39}$$

$$\begin{aligned}
 \mathbf{u}(\xi) &= c^* \mathbf{u}' - s^* (u'_0 \hat{\mathbf{F}}^* - i \hat{\mathbf{F}}^* \times \mathbf{u}') \\
 &= (c^* c - s^* s \hat{\mathbf{F}}^* \cdot \hat{\mathbf{F}}) \mathbf{u} - (2 \operatorname{Re}(c^* s \hat{\mathbf{F}}) + i s^* s \hat{\mathbf{F}}^* \times \hat{\mathbf{F}}) u_0 \\
 &\quad + 2 \operatorname{Im}(c^* s \hat{\mathbf{F}}) \times \mathbf{u} + s^* s (\hat{\mathbf{F}}^* \hat{\mathbf{F}} + \hat{\mathbf{F}} \hat{\mathbf{F}}^*) \cdot \mathbf{u}.
 \end{aligned}
 \tag{40}$$

All terms can now be easily computed,

$$c^* c = \frac{\cosh(\xi E')}{2} + \frac{\cos(\xi B')}{2} \quad s^* s = \frac{\cosh(\xi E')}{2} - \frac{\cos(\xi B')}{2},$$

$$c^*s = \frac{\sinh(\xi E')}{2} + i \frac{\sin(\xi B')}{2} \quad \hat{\mathbf{F}} = \mathbf{e} + i\mathbf{b},$$

$$\mathbf{e} = \frac{1}{\kappa}(E'\mathbf{E} + B'\mathbf{B}) \quad \mathbf{b} = \frac{1}{\kappa}(E'\mathbf{B} - B'\mathbf{E}),$$

$$\hat{\mathbf{F}}^* \cdot \hat{\mathbf{F}} = \frac{E^2 + B^2}{\kappa} \quad \hat{\mathbf{F}}^* \times \hat{\mathbf{F}} = 2i \frac{\mathbf{E} \times \mathbf{B}}{\kappa} \quad \hat{\mathbf{F}}^* \hat{\mathbf{F}} + \hat{\mathbf{F}} \hat{\mathbf{F}}^* = 2 \frac{\mathbf{E}\mathbf{E} + \mathbf{B}\mathbf{B}}{\kappa},$$

$$2 \operatorname{Re}(c^*s\hat{\mathbf{F}}) = \frac{\mathbf{E}}{\kappa}[E' \sinh(\xi E') + B' \sin(\xi B')] + \frac{\mathbf{B}}{\kappa}[B' \sinh(\xi E') - E' \sin(\xi B')],$$

$$2 \operatorname{Im}(c^*s\hat{\mathbf{F}}) = \frac{\mathbf{E}}{\kappa}[E' \sin(\xi B') - B' \sinh(\xi E')] + \frac{\mathbf{B}}{\kappa}[B' \sin(\xi B') + E' \sinh(\xi E')].$$

Putting everything together then yields

$$\begin{aligned} u_0(\xi) = & \left( \frac{\cosh(\xi E')}{2} + \frac{\cos(\xi B')}{2} + \frac{E^2 + B^2}{\kappa} \left[ \frac{\cosh(\xi E')}{2} - \frac{\cos(\xi B')}{2} \right] \right) u_0 \\ & - \left( \frac{\mathbf{E}}{\kappa}[E' \sinh(\xi E') + B' \sin(\xi B')] + \frac{\mathbf{B}}{\kappa}[B' \sinh(\xi E') - E' \sin(\xi B')] \right. \\ & \left. + \frac{\mathbf{E} \times \mathbf{B}}{\kappa}[\cosh(\xi E') - \cos(\xi B')] \right) \cdot \mathbf{u}, \end{aligned} \quad (41)$$

$$\begin{aligned} \mathbf{u}(\xi) = & \left( \frac{\cosh(\xi E')}{2} + \frac{\cos(\xi B')}{2} - \frac{E^2 + B^2}{\kappa} \left[ \frac{\cosh(\xi E')}{2} - \frac{\cos(\xi B')}{2} \right] \right) \mathbf{u} \\ & - \left( \frac{\mathbf{E}}{\kappa}[E' \sinh(\xi E') + B' \sin(\xi B')] + \frac{\mathbf{B}}{\kappa}[B' \sinh(\xi E') - E' \sin(\xi B')] \right. \\ & \left. - \frac{\mathbf{E} \times \mathbf{B}}{\kappa}[\cosh(\xi E') - \cos(\xi B')] \right) u_0 \\ & + \left( \frac{\mathbf{E}}{\kappa}[E' \sin(\xi B') - B' \sinh(\xi E')] + \frac{\mathbf{B}}{\kappa}[B' \sin(\xi B') + E' \sinh(\xi E')] \right) \times \mathbf{u} \\ & + [\cosh(\xi E') - \cos(\xi B')] \frac{\mathbf{E}(\mathbf{E} \cdot \mathbf{u}) + \mathbf{B}(\mathbf{B} \cdot \mathbf{u})}{\kappa}. \end{aligned} \quad (42)$$

The correctness of (41) and (42) can be checked in four ways. First and second, for  $\mathbf{E}=0$  and  $\mathbf{B}=0$ , (41) and (42) reproduces (11)–(13) and (14) and (15), respectively. Third, for  $\mathbf{E} \times \mathbf{B}=0$ , (41) and (42) are identical to (21). Fourth, if initially  $\mathbf{u}=0$  and  $u_0=1$  ( $c=1$ ), then (41) and (42) are in complete agreement with Salingeros's result,<sup>6</sup> his Eq. (28). (Recall that we take  $q=-e$ ,  $\xi=e\tau/m$  here is the negative of Salingeros'  $\zeta=q\tau/m$ .) Thus every term in (41) and (42) has been verified in at least one special case.

## V. CONCLUSIONS

In this work, we have solved the problem of relativistic motion in a constant electromagnetic field by the method of classical evolution operator. The distinct advantage of this approach is that the resulting evolution operator is just the *finite* group generator of the Lorentz group, and the

fundamental group structure of the Lorentz group can be fully exploited for its evaluation. The resulting complex decomposition explains why the solution, despite being real, is fundamentally due to a complex electromagnetic field.

In contrast to other methods, which can only solve the problem when the electromagnetic field is constant, our operator approach can be extended to solve the case when the electromagnetic field is no longer spatially uniform. In this general case, the full evolution operator (7) can still be approximated to any order via

$$W(x_\mu(\tau), u_\mu(\tau)) = \prod_i e^{t_i \tau T} e^{v_i \tau V} W(x_\mu, u_\mu) \quad (43)$$

for suitable sets of coefficients  $\{t_i, v_i\}$ . Since the effects of both  $e^{\tau T}$  and  $e^{\tau V}$  are known, any dynamical variable  $W$  can be evolved forward in  $\tau$  by sequentially updating  $x_\mu$  and  $u_\mu$  via

$$u_\mu(v_i \tau) = e^{v_i \tau V} u_\mu \quad (44)$$

and

$$x_\mu(t_i \tau) = e^{t_i \tau T} x_\mu = x_\mu + t_i \tau u_\mu. \quad (45)$$

The detail of solving relativistic motion in a nonuniform field will be given in a future study.

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