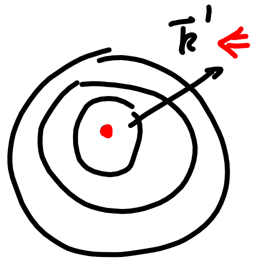

$e^{i\vec{k}\cdot\vec{r}}$




$$\psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\int d^3\vec{r}\,' \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} V(\vec{r}\,')\psi_k(\vec{r}\,')$$

$$\vec{r} >> \vec{r}\,' \qquad |\vec{r}-\vec{r}\,'| = \sqrt{r^2 - 2\vec{r}\cdot\vec{r}\,' + \dots} \approx r\sqrt{1 - 2\frac{\vec{r}}{r^2}\cdot\vec{r}\,'}$$

$$\approx r\left(1 - \frac{\vec{r}}{r^2}\cdot\vec{r}\,' \dots\right) = r - \hat{r}\cdot\vec{r}\,' + \dots$$

$$k|\vec{r}-\vec{r}\,'| \to kr - \underbrace{(k\hat{r})}_{\vec{k}'}\cdot\vec{r}\,'$$

$$\psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \underbrace{\left[-\frac{m}{2\pi\hbar^2}\int d^3\vec{r}\,' e^{-i\vec{k}'\cdot\vec{r}\,'} V(\vec{r}\,')\psi_k(\vec{r}\,')\right]}_{f_k(\hat{r}) = f_k(\Omega_r)} \frac{e^{ikr}}{r}$$

$f_k(\hat{r})$ : $\hat{r} \leftarrow (\theta, \phi)$ = scattering amplitude

**Cross-Section** :

$$d\sigma = \frac{\#\text{ particle scattered into } d\Omega \text{ per unit time}}{\#\text{ incoming particle per unit area per unit time}}$$

$$= \frac{\frac{|f|^2}{\cancel{r^2}}\left(\frac{\hbar k}{m}\right)\cancel{r^2}\, d\Omega}{1\left(\frac{\hbar k}{m}\right)} \Rightarrow \frac{d\sigma}{d\Omega} = |f|^2$$

Born scattering ~ V is weak or $\bar{k} \gg 1$

$$\psi_k(\bar{r}') \sim e^{i\bar{k}\cdot\bar{r}'}$$

$$f_k = -\frac{m}{2\pi\hbar^2}\int d^3\bar{r}'\, e^{-i\bar{k}'\cdot\bar{r}'} V(r')\, e^{i\bar{k}\cdot\bar{r}'}$$

$$= -\frac{m}{2\pi\hbar^2}\langle \bar{k}'|V|\bar{k}\rangle \qquad |\bar{k}'| = |\bar{k}|$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\langle \bar{k}'|V|\bar{k}\rangle\right|^2 \qquad \text{elastic scattering}$$

need to know: $(\bar{k}-\bar{k}')^2 = k^2 + k^2 - 2\bar{k}\cdot\bar{k}'$

$$= 2\cdot 2k^2\left(\frac{1-\cos\theta}{2}\right) = \left[2k\sin\frac{\theta}{2}\right]^2$$

$\hat{r}$, $\theta$, $\bar{k}$

For a Yukawa potential

$$V(r) = V_0\frac{e^{-\alpha r}}{r}$$

nuclear, shielded - Coulomb potential

$$\langle \bar{k}'|V|\bar{k}\rangle = \int d^3\bar{r}'\, V(r')\, e^{i(\bar{k}-\bar{k}')\cdot\bar{r}'}$$

Recall: $$G(r,k) = -\frac{\cancel{m}}{\cancel{\hbar^2}}\frac{1}{2\pi}\frac{e^{ikr}}{r} = \frac{2\cancel{m}}{\cancel{\hbar^2}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{k^2-q^2}e^{i\bar{q}\cdot\bar{r}}$$

$k \to i\alpha$

$$\cancel{-}\frac{1}{2\cancel{\pi}}\frac{e^{-\alpha r}}{r} = \cancel{-}2\int\frac{d^3q}{(2\pi)^2}\frac{1}{\alpha^2+q^2}e^{i\bar{q}\cdot\bar{r}}$$

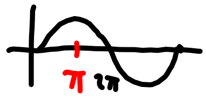
$$\frac{e^{-\alpha r}}{r} = 4\pi\int\frac{d^3q}{(2\pi)^3}\frac{1}{\alpha^2+q^2}e^{i\bar{q}\cdot\bar{r}} \qquad \Delta k = \bar{k}'-\bar{k}$$

$$\langle \bar{k}'|V|\bar{k}\rangle = \frac{4\pi}{\cancel{(2\pi)^3}}\int d^3q\,\frac{1}{\alpha^2+q^2}\underbrace{\int d^3\bar{r}'\, e^{i(\bar{q}-\Delta\bar{k})\cdot\bar{r}'}}_{\cancel{(2\pi)^3}\delta(\bar{q}-\Delta\bar{k})}$$

$$= 4\pi\frac{V_0}{\alpha^2+(\Delta\bar{k})^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{\cancel{4\pi^2}\hbar^4}\frac{\cancel{4\pi}\,\cancel{4\pi}\,V_0^2}{\left[\alpha^2+\left[4k^2\sin^2\frac{\theta}{2}\right]\right]^2}$$

$\sin\frac{\theta}{2}$



$$\frac{d\sigma}{d\Omega} = \frac{V_0^2}{\left[\frac{\hbar^2\alpha^2}{2m} + 4E_k\sin^2\frac{\theta}{2}\right]^2} \xrightarrow[\alpha\to 0]{} \frac{V_0^2}{\left[4E_k\sin^2\frac{\theta}{2}\right]^2}$$

Coulomb scattering

High energy scattering $\rightarrow$ Born

Low " " $\rightarrow$ partial waves

For spherical symmetric potential $\longleftrightarrow$ angular momentum is conserved

$\Rightarrow$ Scattering wave are eigenstate of $\vec{L}$

$\Rightarrow$ The $\ell$th wave of the incoming wave $\longrightarrow$ $\ell$th partial wave of the outgoing wave.

Plane wave decomposition:

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) P_\ell(\cos\theta) \underbrace{J_\ell(kr)}_{\frac{1}{2}\left(h_\ell(kr)+h_\ell^*(kr)\right)}$$

for the exact solution

$$\psi_k(\vec{r}) = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) P_\ell(\cos\theta) R_\ell(kr)$$

$$\left(\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2}\right) R_\ell = \frac{2m}{\hbar^2} V(r)\, r R_\ell$$

at $r \to \infty$ $\quad R_\ell(r) \rightarrow \alpha h_\ell(kr) + \beta h_\ell^*(kr)$

$$\rightarrow B_\ell \left[ h_\ell^*(kr) + S_\ell(\varepsilon) h_\ell(kr) \right]$$

$$\rightarrow \frac{1}{2} \left[ h_\ell^*(kr) + S_\ell(\varepsilon) h_\ell(kr) \right]$$

$h_\ell \sim \frac{e^{ikr}}{r}$

unitarity $\rightarrow$ con. of prob. $\Rightarrow$ $|S_\ell(\varepsilon)|^2 = 1$ $\quad \boxed{S_\ell(\varepsilon) = e^{2i\delta_\ell}}$

$\delta_\ell$ = phase shift

$$\psi_k(r) = \frac{1}{2}\sum_\ell i^\ell (2\ell+1) P_\ell(\cos\theta)\left[h_\ell^*(kr) + e^{2i\delta_\ell} h_\ell(kr)\right]$$

$r \to \infty$

$$+h_\ell(kr) \quad -h_\ell(kr)$$

$$= e^{i\bar{k}\cdot\bar{r}} + \frac{1}{2}\sum_\ell i^\ell (2\ell+1) P_\ell(\cos\theta)\left[e^{i2\delta_\ell} - 1\right] h_\ell(kr)$$

$$h_\ell = \frac{e^{i(kr - \frac{\ell\pi}{2}) - i\frac{\pi}{2}}}{kr}$$

$$= e^{i\bar{k}\cdot\bar{r}} + \frac{1}{2}\sum_\ell \cancel{i^\ell}(2\ell+1) P_\ell \left[e^{i2\delta_\ell} - 1\right]\frac{e^{ikr}}{kri}$$

$$\cancel{(-i)^\ell}$$

$$f = \frac{1}{2ki}\sum_\ell (2\ell+1) P_\ell(\cos\theta)\left[e^{i2\delta_\ell} - 1\right]$$

$$e^{i\delta_\ell}\left[e^{i\delta_\ell} - e^{-i\delta_\ell}\right]$$

$$\boxed{f = \frac{1}{k}\sum_\ell (2\ell+1) P_\ell(\cos\theta)\, e^{i\delta_\ell}\sin\delta_\ell}$$