Angular Momentum Group Properties

Preliminaries : $\vec{L} = \vec{r} \times \vec{p}$ $\quad L_i = \epsilon_{ijk} r_j p_k$ $\quad \epsilon_{ijk} = \begin{cases} 1 & \text{even} \\ -1 & \text{odd} \end{cases}$

i) $\sum_i \epsilon_{ijk} \epsilon_{ij'k'} = \delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{j'k}$ $\quad \Rightarrow \quad \begin{matrix} 3\delta_{kk'} \\ -\delta_{kk'} \end{matrix} = 2\delta_{kk'}$

set $j = j'$ sum over j

$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{C} \cdot A) - \vec{C}(\vec{A} \cdot \vec{B})$

ii) $C\,(A \cdot B) - A\,(B \cdot C)$

Jacobi identity $\quad A\,(B \cdot C) - B\,(\vec{C} \cdot \vec{A})$

$$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0$$

iii) $[A, BC] = ABC - BCA$
$\quad - BAC + BAC$
$\quad = [A,B]C + B[A,C]$

Ex 1 :

$\boxed{[r_i, r_j p_k] = i\delta_{ik}\hbar\, r_j}$

$= [r_i, \cancel{r_j}]p_k + r_j [r_i, p_k]$

Ex 2 :

$[p_i, r_j p_k] = [p_i, r_j]\, p_k + r_j \cancel{[p_i, p_k]}$

$\boxed{[p_i, r_j p_k] = -i\hbar\, \delta_{ij}\, p_k}$

1) $[L_i, r_\ell] = \epsilon_{ijk}[r_j p_k, r_\ell]$
$= \epsilon_{ijk}(-i\hbar\,\delta_{k\ell})r_j$
$= -i\hbar\,\epsilon_{ij\ell} r_j$
$= i\hbar\,\epsilon_{i\ell j} r_j$

Let $\hat{n}$ be a unit vector

$[\vec{L}\cdot\hat{n}, \vec{r}_\ell] = i\hbar\, \hat{n}_i \epsilon_{i\ell j} r_j$
$= i\hbar\, \epsilon_{\ell j i}\, \hat{n}_i r_j$ $= \boxed{i\hbar\, \vec{r} \times \hat{n} = [\vec{L}\cdot\hat{n}, \vec{r}]}$

Similarly $\boxed{[\vec{L}\cdot\hat{n}, \vec{p}] = i\hbar\, \vec{p} \times \hat{n}}$

$$[\hat{n}\cdot\vec{L}, \vec{L}] = [\hat{n}\cdot\vec{L}, \vec{r}\times\vec{p}] = [\hat{n}\cdot\vec{L}, \vec{r}]\times\vec{p} + \vec{r}\times[\hat{n}\cdot\vec{L}, \vec{p}]$$

$$= i\hbar(\vec{r}\times\hat{n})\times\vec{p} + \vec{r}\times(\vec{p}\times\hat{n})\,i\hbar$$

$$= i\hbar\left(\vec{r}\times(\vec{p}\times\hat{n}) + \vec{p}\times(\hat{n}\times\vec{r})\right) \qquad r\to p\to n$$

$$= i\hbar\left(-\hat{n}\times(\vec{r}\times\vec{p})\right)$$

$$[\hat{n}\cdot\vec{L}, \vec{L}] = i\hbar\,(\vec{r}\times\vec{p})\times\hat{n} = i\hbar\,\vec{L}\times\hat{n}$$

$$\hat{n}_i[L_i, L_j] = i\hbar\,\epsilon_{jkl}L_k n_l$$

$$= n_l\, i\hbar\,\epsilon_{ljk}L_k$$

$$\boxed{[L_i, L_j] = i\hbar\,\epsilon_{ijk}L_k}$$

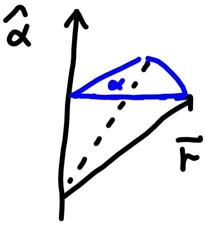
define group property of ang. momentum

or $\qquad \vec{L}\times\vec{L} = i\hbar\,\vec{L}$

Rotation Operator

$$\vec r\,' = \vec r + \vec\alpha \times \vec r$$ ↙ infinitesimal rotation

axis of rotation is given by $\hat\alpha$ and the amount of rotation is α



Since we $i\hbar\, \vec r \times \hat n = [\vec L\cdot\hat n, \vec r]$

$\hat n \to \vec\alpha$

$$-i\hbar(\vec\alpha \times \vec r) = [\vec L\cdot\vec\alpha, \vec r]$$

$$\vec\alpha \times \vec r = \frac{i}{\hbar}[\vec L\cdot\vec\alpha, \vec r]$$

Recall that

$$\vec r\,' = \vec r + \frac{i}{\hbar}[\vec L\cdot\vec\alpha, \vec r] = \left[1+\frac{i}{\hbar}[\vec L\cdot\vec\alpha, \cdot\,] \cdots\right]\vec r$$

$$e^{\epsilon A} B e^{-\epsilon A} = 1+\epsilon[A,B]+\frac{1}{2}\epsilon^2[A,[A,B]]+\cdots$$

$$\boxed{\vec r\,' = e^{\frac{i}{\hbar}\vec\alpha\cdot\vec L}\; \vec r\; e^{-\frac{i}{\hbar}\vec\alpha\cdot\vec L}}$$

For any finite rotation $\vec{\alpha}$, we have

$$\boxed{F' = e^{\frac{i}{\hbar}\vec{\alpha}\cdot\vec{L}}\, F\, e^{-\frac{i}{\hbar}\vec{\alpha}\cdot L}}$$

Since $\frac{i}{\hbar}[\vec{\alpha}\cdot\vec{L}, \vec{F}] = \vec{\alpha}\times\vec{F}$

$$= \left[1 + \frac{i}{\hbar}[\vec{\alpha}\cdot\vec{L}, *] + \frac{1}{2!}(\quad)^2 + \frac{1}{3!}(\quad)^3 \cdots\right]\vec{r}$$

$$= \left[1 + \vec{\alpha}\times + \frac{1}{2!}(\alpha\times)^2 + \frac{1}{3!}(\alpha\times)^3 \cdots\right]\vec{r}$$

$$\vec{r}' = \left[1 + \alpha\hat{\alpha}\times + \frac{1}{2!}\alpha^2(\hat{\alpha}\times)^2 + \frac{\alpha^3}{3!}(\hat{\alpha}\times)^3\right.$$

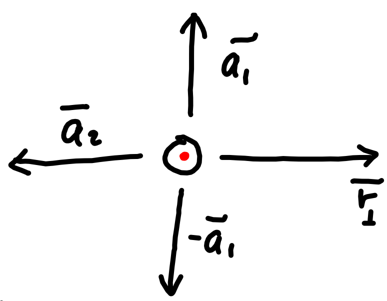
$$\alpha\times(\alpha\times r) \qquad \alpha\times(\alpha\times(\alpha\times r]\ \vec{r}$$

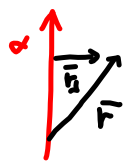
Let $\vec{a}_1 = \hat{\alpha}\times\vec{r}$

$\vec{a}_2 = \hat{\alpha}\times(\hat{\alpha}\times\vec{r})$

$\vec{a}_3 = -\vec{a}_1$

$\vec{a}_4 = -\vec{a}_2$





$$\vec{r}' = \vec{r} + \vec{a}_1\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!}\quad\right)$$
$$+ \vec{a}_2\left(\frac{\alpha^2}{2!} - \frac{\alpha^4}{4!} + \frac{\alpha^6}{6!} \cdots\quad\right)$$

$$= \vec{r} + \vec{a}_1 \sin(\alpha) + (1-\cos\alpha)\vec{a}_2$$

$$\boxed{\vec{r}' = \vec{r} + \hat{\alpha}\times\vec{r}\sin(\alpha) + \hat{\alpha}\times(\vec{\alpha}\times\vec{r})(1-\cos\alpha)}$$

The effect of the rotation operator

$$e^{i\vec{\alpha}\cdot\vec{L}/\hbar}$$

$$\langle \vec{r}_0 | \, e^{i\vec{\alpha}\cdot\vec{L}/\hbar} \, r \, e^{-i\vec{\alpha}\cdot\vec{L}/\hbar} = \langle \vec{r}_0 | \vec{r}\,'$$

$$= \vec{r}_0\,' \langle \vec{r}_0 |$$

$$\underbrace{\langle \vec{r}_0 | \, e^{i\vec{\alpha}\cdot\vec{L}/\hbar}}_{\langle \vec{r}_0 |} \, \vec{r} = \vec{r}_0\,' \underbrace{\langle \vec{r}_0 | \, e^{i\vec{\alpha}\cdot\vec{L}/\hbar}}_{\langle \vec{r}_0\,' |}$$

$$\langle \vec{r}_0 | \, \vec{r} = \vec{r}_0\,' \langle \vec{r}_0\,' |$$

Now if $\quad |\psi'\rangle = e^{i\vec{\alpha}\cdot\vec{L}/\hbar} |\psi\rangle$ rotated wave function

$$\langle \vec{r} | \psi' \rangle = \langle r | e^{i\vec{\alpha}\cdot\vec{L}/\hbar} | \psi \rangle$$

$$= \langle \vec{r}\,' | \psi \rangle$$

rotated wavefunction. $\psi'(\vec{r}) = \psi(\vec{r}\,')$