$$\vec{A} = \frac{1}{me^2}\frac{1}{2}\left[\vec{p}\times\vec{L} - \vec{L}\times\vec{p}\right] - \frac{\vec{r}}{r}$$

Lemma :
1) $\vec{p}\times\vec{L} = -\vec{L}\times\vec{p} + 2i\hbar\vec{p}$
2) $\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$
3) $\vec{p}\times\vec{r} = -\vec{r}\times\vec{p}$
4) $[p_i, \frac{r_i}{r}] = -i\hbar\vec{\nabla}(\frac{\vec{r}}{r}) = -i\hbar\left[\frac{3}{r} - \frac{\vec{r}\cdot\vec{r}}{r^2}\right] = -2i\hbar\frac{1}{r}$

0) $\vec{A}\cdot\vec{L} = 0$ $\quad (\vec{p}\times\vec{L} - \vec{L}\times\vec{p})\cdot\vec{L} - \frac{1}{r}\vec{r}\times\vec{L} \rightarrow \vec{r}\cdot(\vec{r}\times\vec{p}) = 0$

$\vec{p}\cdot\vec{L} = \vec{p}\cdot(\vec{r}\times\vec{p}) = 0$

$(\vec{p}\times\vec{L})\cdot\vec{L}$
$= \vec{p}\cdot(\vec{L}\times\vec{L}) = \vec{p}\cdot\vec{L}$

$\vec{A} = \frac{\gamma}{2}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}) - \frac{\vec{r}}{r}$
$= \gamma(\vec{p}\times\vec{L} - i\hbar\vec{p}) - \frac{\vec{r}}{r}$
$= \gamma(-\vec{L}\times\vec{p} + i\hbar\vec{p}) - \frac{\vec{r}}{r}$

1) $\vec{A}\cdot\vec{A} = \gamma^2(-\vec{L}\times\vec{p} + i\hbar\vec{p})(\vec{p}\times\vec{L} - i\hbar\vec{p})$
$-\gamma\frac{\vec{r}}{r}(\vec{p}\times\vec{L} - i\hbar\vec{p})$

$= \gamma^2\left(-\vec{L}\times\vec{p}\cdot\vec{p}\times\vec{L} + \hbar^2 p^2\right)$ $\quad \left(-\gamma(-\vec{L}\times\vec{p} + i\hbar\vec{p})\frac{\vec{r}}{r} + 1\right.$

$-\gamma\frac{1}{r}\vec{r}\cdot(\vec{p}\times\vec{L}) = -\gamma\frac{1}{r}L^2$ ✓

$\gamma(\vec{L}\times\vec{p})\cdot\vec{r}/r = \gamma\left(\vec{L}\cdot(\vec{p}\times\vec{r})\right)/r$
$= -\gamma L^2/r$ ✓

$= \gamma^2\left(-\epsilon_{ijk}L_j p_k \, p_\ell L_n + \hbar^2 p^2\right)$
$\epsilon_{i\ell n}$
$\hookrightarrow \delta_{j\ell}\delta_{kn} - \delta_{jn}\delta_{\ell k}$

$-\gamma i\hbar\left(\vec{p}\cdot\frac{\vec{r}}{r} - \frac{\vec{r}}{r}\cdot\vec{p}\right) = -2\gamma\hbar^2\frac{1}{r}$ ✓

$-\left(\underbrace{\vec{L}\cdot\vec{p}}_{(\vec{r}\times\vec{p})\cdot\vec{p} = 0}\vec{p}\cdot\vec{L} - L_j p^2 L_j\right) + \hbar^2 p^2$
$- p^2 L^2$

$= \gamma^2 p^2(L^2 + \hbar^2) - \frac{2\gamma^2 L^2}{\gamma r} - \frac{2\gamma^2\hbar^2}{\gamma r} + 1$

$= -2\frac{\gamma^2}{\gamma r}[L^2 + \hbar^2] + 1$ $\quad \gamma = \frac{1}{me^2}$

$= \gamma^2(L^2 + \hbar^2)\left(p^2 - \frac{2}{\gamma r}\right) + 1$
$\left(p^2 - \frac{2me^2}{r}\right)$ $= \gamma^2(L^2 + \hbar^2)2m\left(\frac{p^2}{2m} - \frac{e^2}{r}\right) + 1$

1) $\boxed{A^2 = \gamma^2(L^2 + \hbar^2)2mH + 1}$ $\quad$ 2) $[H, \vec{A}] = 0$

3) $[A_i, A_j] = i\hbar \frac{-2H}{me^4} \epsilon_{ijk} L_k$!

define $K_i = \sqrt{\frac{-me^4}{2H}}\, A_i$ $\quad$ K_i is hermitian only if $H < 0$

$$[K_i, K_j] = i\hbar\, \epsilon_{ijk} L_k$$
$$[L_i, K_j] = i\hbar\, \epsilon_{ijk} K_k$$
$$[L_i, L_j] = i\hbar\, \epsilon_{ijk} L_j$$

closed algebra

$$-\frac{2HK^2}{me^4} \;/\!/\; A^2 = \frac{1}{m^{\cancel{2}}e^4}(L^2+\hbar^2)\,2\cancel{m}H + 1$$

$$-1 = \frac{(L^2+K^2+\hbar^2)\,2H}{me^4}$$

or

$$\boxed{H = -\frac{me^4}{2(L^2+K^2+\hbar^2)}}$$

$$\frac{1}{a_0} = \frac{me^2}{\hbar^2}$$

$$Ha:\ \frac{e^2}{a_0} = \frac{me^4}{\hbar^2}$$

what is $\vec{K}$?

define $\vec{M} = \frac{1}{2}(\vec{L}+\vec{K})$ $\quad \vec{N} = \frac{1}{2}(\vec{L}-\vec{K})$

$$[M_i, M_j] = \frac{1}{4}[L_i + K_i, L_j + K_j] = i\hbar \epsilon_{ijk} M_k$$

$$[N_i, N_j] = i\hbar \epsilon_{ijk} N_k \qquad [N_i, M_j] = 0 \;!$$

eigenvalues of M $\quad \mathcal{M}(\mathcal{M}+1)\hbar^2$ $\quad (2\mathcal{M}+1)$ m states

N $\quad \mathcal{N}(\mathcal{N}+1)\hbar^2$ $\quad (2\mathcal{N}+1)$ n states $\quad \vec{A}\cdot\vec{L} = 0$

$\mathcal{M}, \mathcal{N} \to 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

$$\vec{M} = \frac{1}{2}(\vec{L}+\vec{K})$$

$$H = -\frac{me^4}{2(K^2+L^2+\hbar^2)} \qquad M^2 = \frac{1}{4}(L^2 + \underbrace{\vec{L}\cdot\vec{K} + \vec{K}\cdot\vec{L}}_{=0} + K^2)$$

$$N^2 = \frac{1}{4}(L^2+K^2)$$

$$L^2 + K^2 = 4M^2 = 4\mathcal{M}(\mathcal{M}+1)\hbar^2$$

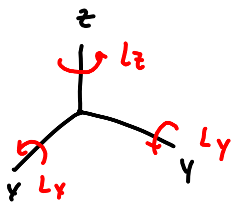
$$H = -\frac{me^4}{2\hbar^2(4\mathcal{M}^2+4\mathcal{M}+1)} \qquad N^2 = M^2$$

$2\mathcal{M}+1 = 1, 2, 3, 4, \dots$

In 3D

$$= -\frac{me^4}{2\hbar^2(2\mathcal{M}+1)^2} = -\frac{1}{2(2\mathcal{M}+1)^2} H_a$$

$2\mathcal{M}+1 = n$



L_z : xy plane $\qquad = -\frac{1}{2n^2} H_a$

L_y = yz plane

L_x = zx plane $\qquad$ one more dimension : w

M "

rotation in the $\left.\begin{array}{l} wx \\ wy \\ wz \end{array}\right.$ plane $\quad$ 4 D

3 more $\{$ angular momenta !

$\Rightarrow \vec{L}_\pm \to$ make $(2m+1)$ m state degenerate

$\Rightarrow \vec{A}$ $\qquad n = 1 \quad \ell = 0$

$= 2 \quad \ell = 0, \ell = 1$

$= 3 \quad \ell = 0, \ell = 1, \ell = 2$

For the Kepler problem : orbit is position space is an ellipse

" momentum " is just a circle

momentum space