Take $\mu \sim m_e = m$

$$\left(-\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{r}\right)\psi = E\psi$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

choose a, a length scale

$x = x^* a$
$y = y^* a$
$z = z^* a$

$\boxed{\frac{\hbar^2}{ma^2}}$ is an energy

$$-\frac{\hbar^2}{2ma^2}\nabla_*^2$$

all energy in units of $\frac{\hbar^2}{ma^2}$.

$\frac{\hbar^2 k^2}{m}$

$$\left(-\frac{\hbar^2}{2ma^2}\nabla_*^2 - \frac{e^2}{r^* a}\right)\psi = E\psi$$

$\because E = E^* \frac{\hbar^2}{ma^2}$

$$\left(-\frac{1}{2}\nabla_*^2 - \frac{e^2}{\cancel{a}}\frac{ma^{\cancel{2}}}{\hbar^2}\frac{1}{r^*}\right)\psi = E^*\psi$$

choose a such that $\frac{e^2 m a}{\hbar^2} = 1 \implies a =$ Bohr's radius

$$a = \frac{\hbar^2}{me^2}$$

$$\boxed{\left(-\frac{1}{2}\nabla_*^2 - \frac{1}{r^*}\right)\psi = E^*\psi}$$

$E^* = -\frac{1}{2}\frac{1}{n^2}$

$$\frac{\hbar^2 e^2}{e^2 m a^2} = \frac{a e^2}{a^2} = \frac{e^2}{a}$$

"Hartree" Ha

$E^* = -\frac{1}{2}\frac{1}{n^2}$ Ha

$\frac{1}{2} E_h = 13.6\,\text{eV}$

$E_h = 2(13.6)\,\text{eV}$

For the case of HO

$$-\frac{1}{2}\frac{\hbar^2}{m}\frac{d^2}{dx^2}\psi + \frac{m}{2}\frac{k}{m}x^2\psi = E\psi$$

$$\frac{k}{m} = \omega^2$$

$$-\frac{1}{2}\frac{\hbar^2}{m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi$$

$x = x^* a$

$$-\frac{1}{2}\frac{\hbar^2}{\cancel{m}a^2}\frac{d^2}{dx^{*2}} + \frac{1}{2}m\omega^2a^2x^{*2}\psi = E^*\frac{\hbar^2}{\cancel{m}a^2}\psi$$

$$-\frac{1}{2}\frac{d^2}{dx^{*2}} + \frac{1}{2}m\omega^2a^2\frac{ma^2}{\hbar^2}x^*\psi = E^*\psi$$

choose a such that

$$m^2\omega^2a^4 = \hbar^2$$

$$a = \sqrt{\frac{\hbar}{m\omega}} \quad \text{harmonic length}$$

$$x^*\sqrt{\frac{\hbar}{m\omega}} = x = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^+) \qquad p = -i\hbar\frac{\partial}{\partial x}$$

$$\Rightarrow \quad x^* = \frac{1}{\sqrt{2}}(a+a^+) \qquad p^*\frac{\hbar}{a} = -i\hbar\frac{\partial}{a\,\partial x^*}$$

$$i\hbar = [x,p] = [x^*,p^*]\hbar = i\hbar \qquad p^* = -i\frac{\partial}{\partial x^*}$$

$$[x^*,p^*] = i$$

In dimensionless form (in atomic unit)

$r^* = r$

$$\left(-\frac{1}{2}\frac{d^2}{dr^2} + \frac{l(l+1)}{2r^2} - \frac{1}{r}\right)u = E\,u \qquad u = rf$$

$$u(r) = r^n e^{-r/n} = e^{n\ln r - r/n} \quad \leftarrow \text{exponential}$$

$$u' = \left(\frac{n}{r} - \frac{1}{n}\right)u$$

$$u'' = \left[-\frac{n}{r^2} + \left(\frac{n}{r} - \frac{1}{n}\right)^2\right]u$$

$$= \left[-\frac{n}{r^2} + \frac{n^2}{r^2} - \frac{2}{r} + \frac{1}{n^2}\right]u$$

$$\left(-\frac{1}{2}\frac{n(n-1)}{r^2} + \cancel{\frac{1}{r}} - \frac{1}{2}\frac{1}{n^2} \qquad \frac{n(n-1)}{r^2}\right.$$
$$\left.\frac{l(l+1)}{2r^2} - \cancel{\frac{1}{r}}\right)\psi = E\,\psi$$

$$\boxed{\begin{array}{l}\text{Solution is} \\ n = l+1 \\ E = -\frac{1}{2}\frac{1}{n^2}\end{array}}$$

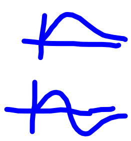
For a given ℓ, always have

Solution

reduced radial $u = r^{\ell+1} e^{-r/(\ell+1)} = r f(r)$

radial $\boxed{f(r) = r^{\ell} e^{-r/\ell+1}}$ $n = \ell + 1$

# degeneracy	n			
1	$n = 1$	$\ell = 0$	$f = e^{-r}$	$E = -\frac{1}{2}\,Ha$
$1 + 3 = 4 = 2^2$	2	$\ell = 1$, $\ell = 0$	$f = r e^{-r/2}$	$= -\frac{1}{2}\frac{1}{4}\,Ha$
$1+3+5 = 9 = 3^2$	3	$\ell = 2$, $\ell = 1$, $\ell = 0$	$f = r^2 e^{-r/3}$	
$1+3+5+7 = 16 = 4^2$	4	$\ell = 3$, $\ell = 2$, $\ell = 1$, $\ell = 0$		



For a given energy level

$E = -\frac{1}{2}\frac{1}{n^2}\,Ha$

n^2 - degeneracy

The hidden symmetry of Hydrogen

a conserved quantity ⇒ a const of motion.

Laplace - Runge - Lenz vector: classically

$$\vec{A} = \frac{1}{me^2}\vec{p}\times\vec{L} - \frac{\vec{r}}{r} \quad \leftarrow \text{usual definition}$$

$|\vec{A}| =$ "e" eccentricity of the classical orbit

Conservations of

E, $\vec{L}$, $A(\theta)$

1 3 1 ← without

for 3D orbit phase space is 6D

deg. f = 6 $(\vec{r},\vec{p})$

phase-space = 6 − 4 = 2 "space filling" orbit

with 6 − 5 = 1 ← closed orbit.

Show that $\dot{\vec{A}} = 0$

$$= \gamma\dot{\vec{p}}\times\vec{L} - \left(\dot{\frac{\vec{r}}{r}}\right) \qquad \gamma = \frac{1}{me^2}$$

$$\left(\dot{\frac{\vec{r}}{r}}\right) = \frac{\dot{\vec{r}}}{r} - \frac{\vec{r}}{r^2}\dot{r} = \frac{1}{r^2}(r\dot{\vec{r}} - \dot{r}\vec{r})$$

$$\vec{r}\times(\vec{r}\times\dot{\vec{r}})$$

compare

$$(\vec{r}\cdot\dot{\vec{r}})\vec{r} - r^2\dot{\vec{r}}$$

$\vec{r}^2 = r^2$

$2\vec{r}\cdot\dot{\vec{r}} = 2r\dot{r}$

$$r\dot{r}\vec{r} - r^2\dot{\vec{r}}$$

$$r(\dot{r}\vec{r} - r\dot{\vec{r}})$$

$$= -\frac{1}{r^3}\vec{r}\times(\vec{r}\times\dot{\vec{r}})$$

$$\dot{\vec{A}} = \frac{1}{me^2}\left(-\frac{e^2}{r^2}\hat{r}\right)\times L \qquad V = -\frac{e^2}{r}$$

$$+\frac{1}{mr^3}\vec{r}\times(\vec{r}\times m\dot{\vec{r}}) \qquad \dot{p} = -\frac{e^2}{r^2}\hat{r}$$

$= 0$!

What is $\vec{A}$?

$$\vec{A} = \frac{1}{me^2}\vec{p}\times\vec{L} - \frac{\vec{r}}{r}$$

$$\vec{A}\cdot\vec{r} = Ar\cos\theta$$

$$\frac{1}{me^2}\vec{r}\cdot(\vec{p}\times\vec{L}) - r = Ar\cos\theta$$

$$(\vec{r}\times\vec{p})\cdot\vec{L}$$

$$\frac{1}{me^2}L^2 = r[1 + A\cos\theta]$$

$$\Rightarrow \quad r = \frac{L^2}{me^2}\frac{1}{1+A\cos\theta}$$

elliptical orbit with eccentricity "A"

$\vec{A}$ is vector with mag = "e"

and points along the semi-major axis of the ellipse.

the constancy of $\vec{A}$, the ellipse will not precess!

any none $1/r^2$ force will cause the orbit to ↑

classically

$$\vec{A} = \frac{1}{me^2}\vec{p}\times\vec{L} - \frac{\vec{r}}{r}$$

$$\vec{A}^2 = \frac{1}{(me^2)^2}(\vec{p}\times\vec{L})^2 - 2\frac{1}{me^2}\underbrace{\frac{\vec{r}}{r}\cdot\vec{p}\times\vec{L}} + 1$$

$$= \frac{1}{(me^2)^2}p^2L^2$$

since $\vec{p}\perp\vec{L}$

$$\frac{1}{r}\vec{r}\cdot(\vec{p}\times\vec{L})$$

$$\frac{1}{r}\vec{r}\times p\cdot L$$

$$\frac{1}{r}L^2$$

$$= \frac{1}{(me^2)^2}p^2L^2 - \frac{2}{me^2}\frac{L^2}{r} + 1$$

$$= \frac{1}{(me^2)^2}L^2\left[p^2 - \frac{me^2\,2}{r}\right]$$

$$A^2 = \frac{1}{(me^2)^2}L^2(2m)\underbrace{\left[\frac{p^2}{2m} - \frac{e^2}{r}\right]}_{H} + 1$$

$$A^2 = \frac{1}{(me^2)^2}L^2\,2mH + 1 = \epsilon^2$$

$|\vec{A}|$ not a new constant of motion.

Quantum Mechanically : The hermitian form of A is

$$\vec{A} = \frac{1}{me^2}\frac{1}{2}\left[\vec{p}\times L - \vec{L}\times\vec{p}\right] - \frac{\vec{r}}{r}$$