The needed operator is

$$\boxed{\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}\, r - \frac{1}{r^2}\frac{L^2}{\hbar^2}}$$

L^2 only involves θ, ϕ

To show this $L^2 = (\vec{r}\times\vec{p})\cdot(\vec{r}\times\vec{p})$

$$= \epsilon_{ijk}\, r_j p_k\, \epsilon_{ij'k'}\, r_{j'} p_{k'}$$

$$= (\delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'})\, r_j p_k r_{j'} p_{k'}$$

$$= r_j \underbrace{p_k r_j}_{r_j p_k - [r_j, p_k]} p_k - r_j p_k \underbrace{r_k p_j}_{p_j r_k + [r_k, p_j]}$$

$[r_j, p_k] = -i\hbar\,\delta_{jk}$ (first term $\rightarrow$ $\vec{r}^2\vec{p}^2 - i\hbar\,\vec{r}\cdot\vec{p}$); $\quad p_j r_k + i\hbar\,\delta_{kj}$

$$'' \quad -r_j p_k p_j r_k - i\hbar\,\vec{r}\cdot\vec{p}$$

$$-r_j p_j \underbrace{p_k r_k}_{[p_k, r_k] + r_k p_k}$$

$$'' \quad -(\vec{r}\cdot\vec{p})^2 + i\hbar\, 3\, \vec{r}\cdot\vec{p}$$

$$= \vec{r}^2\vec{p}^2 - i\hbar\,\vec{r}\cdot\vec{p} - (\vec{r}\cdot\vec{p})^2 + 2i\hbar\,\vec{r}\cdot\vec{p}$$

$$L^2 = r^2\vec{p}^2 - (\vec{r}\cdot\vec{p})^2 + i\hbar\,\vec{r}\cdot\vec{p}$$

$$\frac{L^2}{r^2} = \vec{p}^2 - \frac{1}{r^2}(\vec{r}\cdot\vec{p})^2 + i\hbar\frac{1}{r^2}(\vec{r}\cdot\vec{p})$$

$$\frac{L^2}{\hbar^2 r^2} = -\cancel{\hbar^2}\nabla^2 + \cancel{\hbar^2}\frac{1}{r^2}\left(r\frac{\partial}{\partial r}\right)^2 + \cancel{\hbar^2}\frac{1}{r^2}\left(r\frac{\partial}{\partial r}\right)$$

$$\nabla^2 = -\frac{L^2}{\hbar^2 r^2} + \frac{1}{r^2}\left(r\frac{\partial}{\partial r}\right)\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}$$

$$= -\frac{L^2}{\hbar^2 r^2} + \underbrace{\frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}}_{\left(\frac{1}{r}\frac{\partial^2}{\partial r^2} r\right)}$$

$$\frac{\partial^2}{\partial r^2}(r f) \quad + 2\cancel{\frac{\partial r}{\partial r}}\frac{\partial f}{\partial r} + r\frac{\partial^2}{\partial r^2}$$

$$\boxed{\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}\, r - \frac{L^2}{\hbar^2 r^2}}$$

Eigenvalues + functions of L

Since $[\vec{L}, L^2] = 0$, $[L_x, L_x^2 + L_y^2 + L_z^2]$

$= [L_x, L_y^2] + [L_x, L_z^2]$

$= [L_x, L_y]L_y + [L_x, L_z]L_z + L_y[L_x, L_y] + L_z[L_x, L_z]$

$= i\hbar[L_z L_y + L_y L_z] - i\hbar[L_y L_z + L_z L_y]$

$= 0$

Can take the eigenvalues of L^2 + L_z simultaneously

$\boxed{m\hbar}$ $\quad$ $\boxed{\ell(\ell+1)\hbar^2}$

Since $\langle\psi| L^2 |\psi\rangle > 0$

Consider $[L_z, L_\pm] =$ $\qquad L_\pm = L_x \pm i L_y$

$= [L_z, L_x] \pm i[L_z, L_y]$

$= i\hbar L_y \pm i(-i\hbar L_x)$ $\qquad [L_i, L_j] = i\hbar \epsilon_{ijk} L_k$

$= \pm\hbar(L_x \pm i L_y) = \pm\hbar L_\pm$

$L_z L_\pm = L_\pm L_z \pm \hbar L_\pm$ $\qquad \mathcal{L}_+$

$L_z L_\pm |\ell, m\rangle = L_\pm(m\hbar \pm \hbar)|\ell, m\rangle$ $\qquad |\ell, m\rangle$

$= (m\hbar \pm \hbar) L_\pm |\ell, m\rangle$

$L_\pm |\ell, m\rangle = c\, |\ell, m \pm 1\rangle$

Since $(L_-)^\dagger = L_+$ $\qquad \langle \ell, m| L_\mp L_\pm |\ell, m\rangle = c^2$

$L_+ L_- = (L_x + i L_y)(L_x - i L_y)$

$= L_x^2 + i L_y L_x - i L_x L_y + L_y^2$

$= L^2 - L_z^2 - i\, i\hbar L_z = L^2 - L_z^2 + \hbar L_z$

$L_- L_+ = L^2 - L_z^2 - \hbar L_z$

$\langle \ell m| L_\mp L_\pm |\ell m\rangle = \ell(\ell+1)\cancel{\hbar^2} - m^2\cancel{\hbar^2} \mp m\cancel{\hbar^2} \geq 0$

$\ell(\ell+1) \geq m(m \pm 1)$

This means that $\qquad m = \ell \checkmark \qquad -\ell(-\ell-1)$

$\Rightarrow \qquad m = -\ell \checkmark \qquad = \ell(\ell+1)$

$m = -\ell, -\ell+1, -\ell+2 \ldots 0, \ldots \ell$

$L_\pm |\ell, m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m\pm1)}\; |\ell, m \pm 1\rangle$

Allows for $\ell = \frac{1}{2}$ spin $\frac{1}{2}$ $\qquad m = -\frac{1}{2}, \frac{1}{2}$

$= 1 \qquad m = -1, 0, 1$

etc $= \frac{3}{2}$

Spherical Harmonics, eigenfunctions of L^2, L_z

$$\langle \theta, \phi | \ell m \rangle = Y_{\ell m}(\theta, \phi)$$

$Y_{00} = \frac{1}{\sqrt{4\pi}}$

$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$

etc.

$$\underbrace{\int d\Omega}_{\int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi} \; Y^*_{\ell' m'}(\theta, \phi) \, Y_{\ell m}(\theta, \phi) = \delta_{\ell\ell'} \delta_{mm'}$$

$$L^2 Y_{\ell m} = \ell(\ell+1)\hbar^2 Y_{\ell m}$$

$$L_z Y_{\ell m} = m\hbar Y_{\ell m}$$

generalization of Fourier Series $[0, 2\pi]$ to a spherical surface

$$f(\theta, \phi) = \sum_{\ell, m} f_{\ell m} Y_{\ell m}(\theta, \phi) \longrightarrow f(\phi) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\phi} f_\ell$$

The general central force problem

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(r)$$

spherically symmetric

$[H, L\pm] = 0$ $[H, \vec{L}] = 0$ $[H, L^2] = 0$

$$H(L\pm|\psi\rangle) = L\pm H|\psi\rangle$$
$$= L\pm E|\psi\rangle$$
$$= E(L\pm|\psi\rangle)$$

all states of m-values have identical energy

Can choose (E, ℓ, m) as the complete set of eigenfunctions

Since $L\pm|E,\ell,m\rangle = |E,\ell,m\pm1\rangle$

all $2\ell+1$ m-states are degenerate.

For orbital angular momentum

$$\langle r, \theta, \phi | l m \rangle = \langle r, \theta, \phi + 2\pi | l m \rangle$$

$$= \langle r, \theta, \phi | e^{2\pi i L_z/\hbar} | l m \rangle$$

$$= \langle r, \theta, \phi | e^{2\pi i m} | l m \rangle$$

then m = integer $\Rightarrow \pm l$ are also integers

From our previous result

$$\nabla^2 = -\frac{1}{r^2}\frac{1}{\hbar^2}L^2 + \frac{1}{r}\frac{\partial^2}{\partial r^2}r$$

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{2m}\frac{L^2}{r^2} + V(r)\right]\psi(r) = E\psi(r)$$

$$\text{if } \psi(r,\theta,\phi) = R(r)Y_{\ell m}(\theta,\phi) \rightarrow f_\ell(r)Y_{\ell m}(\theta,\phi)$$

$$L^2 \rightarrow \ell(\ell+1)\hbar^2$$

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2} + V(r)\right]f_\ell(r) = E f_\ell(r)$$

define $\boxed{u_\ell(r) = r f_\ell(r)}$

$$\underline{\left[-\frac{\hbar^2}{2m}\cancel{\frac{1}{r}}\frac{\partial^2 u_\ell}{\partial r^2} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2} + V(r)\right]u_\ell(r) = E u_\ell(r)}$$

radial Schrodinger Eq.

General properties of $u_\ell(r)$ $\qquad \dfrac{u_\ell}{r} = f_\ell$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 \ell(\ell+1)}{2m\, r^2} + V(r)\right) u_\ell = E\, u_\ell(r)$$

1) $V(r)$ is not more singular than $-\frac{1}{r^2}$

2) as $r \to 0$, neglect $V(r)$

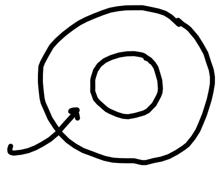
$$-\frac{d^2}{dr^2} u_\ell + \frac{\ell(\ell+1)}{r^2} u_\ell = 0 \qquad \text{take } u_\ell = r^n$$

$$-n(n-1) r^{n-2} + \ell(\ell+1) r^{n-2} = 0$$

$$n(n-1) = \ell(\ell+1) \implies \left.\begin{array}{l} n = \ell+1 \\ \text{or } n = -\ell \end{array}\right\} \text{two solutions}$$

$u_\ell \sim r^{\ell+1}$ regular

$\sim r^{-\ell}$ irregular allowed in



4) $r \to \infty$ $\quad V(r) \to 0$

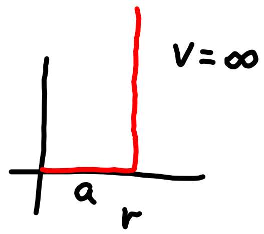
$$-\frac{\hbar^2}{2m}\frac{d^2 u_\ell}{dr^2} = E\, u_\ell$$

$$\frac{d^2 u_\ell}{dr^2} = -\frac{2mE}{\hbar^2} u_\ell \qquad e^{\pm ikr} \text{ continuum state}$$

$$= e^{\pm \alpha r} \text{ bound state}$$

"k^2" $\qquad k = \sqrt{\frac{2mE}{\hbar^2}}$

The spherical well :

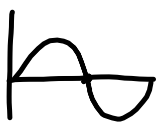


$$\frac{d^2 u_\ell}{dr^2} + k^2 u_\ell - \frac{\ell(\ell+1)}{r^2} u = 0$$

for $\ell = 0$ $\quad u_\ell \sim e^{\pm ikr}$ or $\sin(kr), \cos(kr)$

$$f_\ell(r) = \frac{u_\ell}{r} \simeq \frac{1}{r} e^{\pm ikr} \text{ or } \frac{\sin(kr)}{r}, \frac{\cos(kr)}{r}$$

In order for $f_\ell(r=a) = 0$



$$\underset{=0}{f_\ell(r)} = A \frac{\sin(kr)}{r}$$

$ka = n_r \pi$ $\quad n_r = 1, 2, 3$ radial q. number

$k_n = \frac{n_r \pi}{a}$

$$E_{\underset{0}{\overset{\shortparallel}{\ell}} n_r} = \frac{\hbar^2}{2m}\left(\frac{n_r \pi}{a}\right)^2 \rightarrow \text{depends on } \ell, n_r$$

For $\ell \neq 0$, the solutions for $f_\ell(r)$ are

Spherical Bessel function

$x = kr$

For inf. spherical well

$$j_0(x) = \frac{\sin x}{x} \qquad n_0(x) = -\frac{\cos x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$j_\ell(k_{\ell n_r} a) = 0 \qquad \boxed{k_{\ell n_r} = \frac{x_{\ell, n_r}}{a}} \leftarrow \text{the zeros of spherical Bessel fctions.}$$

$$j_\ell(x_{\ell, n_r}) = 0 \qquad \boxed{E_{\ell n_r} = \frac{\hbar^2}{2m} k_{\ell n_r}^2}$$

↑ 1st, second, third zero, etc.

↳ for each ℓ .
(2ℓ+1) m states are degenerate.

Properties of spherical Bessel fct :

$x = kr$

$x \to 0 \quad j_\ell(x) \to \dfrac{x^\ell}{1\cdot3\cdot5\ (2\ell+1)} \qquad \eta_\ell(x) = -\dfrac{1\cdot1\cdot3\cdot5\ (2\ell-1)}{x^{\ell+1}}$

$u_\ell \sim r^{\ell+1}$

$f_\ell = \frac{1}{r} u_\ell \sim r^\ell$

$$j_\ell(x) \underset{x\to\infty}{\to} \frac{1}{x}\cos\left(x - \frac{(\ell+1)\pi}{2}\right) = \frac{1}{x}\sin\left(x - \frac{\ell\pi}{2}\right)$$

$$\eta_\ell \underset{x\to\infty}{\longrightarrow} \frac{1}{x}\sin\left(x - \frac{(\ell+1)\pi}{2}\right) = -\frac{1}{x}\cos\left(x - \frac{\ell\pi}{2}\right)$$

Hankel fct

$$h_\ell = j_\ell + i\eta_\ell = \frac{1}{x} e^{i\left(x - \frac{(\ell+1)\pi}{2}\right)}$$

$$h_\ell^* = j_\ell - i\eta_\ell = \frac{1}{x} e^{-i\left(x - \frac{(\ell+1)\pi}{2}\right)}$$