$$\hat{A}(t) = e^{itH/\hbar}\,\hat{A}(0)\,e^{-itH/\hbar} = e^{\epsilon H}\hat{A}\,e^{-\epsilon H} = A(\epsilon)$$

$$i\hbar\frac{\partial}{\partial t}\hat{A}(t) = [\hat{A}(t), H] \qquad \epsilon = \frac{it}{\hbar}$$

$$i\hbar\,\frac{i\,\partial}{\hbar\,\partial\epsilon}\hat{A} \qquad t = \frac{\epsilon\hbar}{i}$$

$$\frac{\partial\hat{A}}{\partial\epsilon} = [H, \hat{A}(\epsilon)]$$

$$= [H, *]\,\hat{A}$$

$$\hat{A}(\epsilon) = e^{\epsilon[H, *]}A$$

$$= \left[1 + \epsilon[H, *] + \tfrac{1}{2}\epsilon^2[\ \ ]^2 + \cdots\right]A$$

$$e^{\epsilon H}A\,e^{-\epsilon H} = A + \epsilon[H, A] + \tfrac{1}{2}\epsilon^2[H,[H,A]] + \tfrac{1}{3!}\epsilon^3[H[H[H,A]]] + \cdots$$

for the case of HO

$$A = x \qquad \hat{x}(t) = e^{\epsilon H}\,\hat{x}\,e^{-\epsilon H} \qquad \epsilon = \frac{it}{\hbar} \quad \epsilon^2 = -\frac{t^2}{\hbar^2}$$

$$\hat{x}(t) = x + \epsilon[\hat{H},x] + \frac{1}{2}\epsilon^2\,[\hat{H},[\hat{H},x]] + \ldots \qquad \hat{x} = \hat{x}(0)$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \qquad [H,\hat{x}] = -i\hbar\frac{\hat{p}}{m} \qquad [H,\hat{p}] = i\hbar m\omega^2\hat{x}$$

$$[H^2,x] \equiv [H,[H,x]] = [H,\hat{p}]\left(-i\frac{\hbar}{m}\right) = \left(-i\frac{\hbar}{m}\right)(i\hbar m\omega^2)\,\hat{x}$$

$$[H^{2n},x] = (\hbar\omega)^{2n}\,x \qquad = \hbar^2\omega^2\hat{x}$$

$$[H^2,p] = i\hbar m\omega^2[H,x] = i\hbar m\omega^2\left(-i\hbar\frac{\hat{p}}{m}\right) = \hbar^2\omega^2\hat{p}$$

$$[H^{2n},p) = (\hbar\omega)^{2n}\hat{p}$$

$$\hat{x}(t) = x + \frac{1}{2}\left(-\frac{t^2}{\hbar^2}\right)\hbar^2\omega^2 x + \frac{1}{4!}\left(-\frac{t^2}{\hbar^2}\right)^2(\hbar^2\omega^2)^2 x$$

$$= \left(1 - \frac{1}{2}t^2\omega^2 + \frac{1}{4!}t^4\omega^4 \ldots\right)x = \cos(\omega t)\,\hat{x}$$

$$+ \frac{1}{m\omega}\hat{p}\sin(\omega t)$$ ⇐ do this

Time - dependent Hamiltonian

Simple case : $H = \frac{p^2}{2m} + V(\vec{r}) - \vec{F}(t)\cdot\vec{r}$

dipole - interaction

$\vec{F}(t) = e\vec{E}_0 \sin(\omega t)$

Seek solution of the form

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle$$

$$i\hbar \frac{\partial}{\partial t} U(t) = \hat{H}_S U(t)$$

$\Rightarrow U^\dagger U = 1$ — evolution op.

Hamiltonian in the Schrodinger picture.

For time - independ : $U(t) = e^{-i\frac{t}{\hbar}H_S}$

For the case of $H_S = H_S(t)$, in terms of U

$$\langle \hat{A}(t)\rangle = \langle \psi(t)|A|\psi(t)\rangle = \langle\psi(0)|\underbrace{U^\dagger A U}_{\hat{A}(t)}|\psi(0)\rangle$$

$$i\hbar\dot{\hat{A}} = i\hbar\dot{U}^\dagger A U + U^\dagger A\, i\hbar\dot{U}$$

$$= U^\dagger[-H_S A + A H_S]U$$

$$i\hbar\dot{\hat{A}} = U^\dagger[A, H_S]U$$

$i\hbar\dot{\hat{A}} = [A, H_H(t)]$

For example:

$$i\hbar\dot{\hat{r}} = U^\dagger[\hat{r},\hat{p}^2]\frac{1}{2m}U = \frac{i\hbar}{m}\hat{p}(t)$$

$$i\hbar\dot{\hat{p}} = U^\dagger[\hat{p}, V(\hat{r})]U = -\vec{\nabla}V + \vec{F}(t)$$

The general case of time-dependent Hamiltonian

→ with non-commuting operators.

$$i\hbar \frac{dU}{dt} = H_s(t)\,U(t)$$

$$U(t) = e^{-\frac{i}{\hbar}\int^t H_s(t')\,dt'}\,U(0) \qquad U(0) \to 1$$

if $H_s(t)$ is just a fct of t

However, if $[H_s(t'), H_s(t)] \neq 0$

→ contain non-commuting op

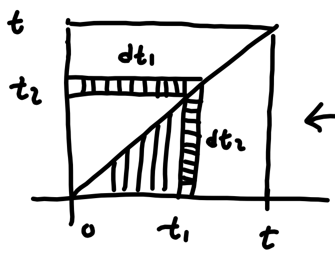
$$i\hbar \int_0^t \frac{dU}{dt} = \int_0^t H_s(t_1)\,U(t_1)\,dt_1$$

$$i\hbar \left[U(t) - U(0) \right] = \int_0^t H_s(t_1)\,U(t_1)\,dt_1 \qquad U(0) = 1$$

$$U(t) = 1 + \frac{1}{i\hbar}\int_0^t H_s(t_1)\,U(t_1)\,dt_1$$

$$U(t) = 1 + \frac{1}{i\hbar}\int_0^t H_s(t_1)\,dt_1 + \frac{1}{(i\hbar)^2}\int_0^t H_s(t_1)\int_0^{t_1} H_s(t_2)\,dt_2\,dt_1 \qquad (U(t_2))$$

$$= 1 + \frac{1}{i\hbar}\int_0^t H_s(t_1)\,dt_1 + \frac{1}{(i\hbar)^2}\underbrace{\int_0^t dt_1 \int_0^{t_1} dt_2\, H_s(t_1)\,H_s(t_2)}_{I} + \cdots$$



$t_1 > t_2$ — I

$t_1 \leftrightarrow t_2$: $\int_0^t dt_2 \int_0^{t_2} dt_1\, H_s(t_2)\,H_s(t_1)$, $t_2 > t_1$

$$= 1 + \frac{1}{i\hbar}\int_0^t H_s(t_1)\,dt_1 + \frac{1}{(i\hbar)}\,\frac{1}{2}\int_0^t dt_2 \int_0^t dt_1\, T\left[H(t_1)\,H(t_2)\right]$$

$$T\left[H(t_1)\,H(t_2)\right] = \begin{cases} H(t_1)\,H(t_2) & \text{if } t_1 > t_2 \\ H(t_2)\,H(t_1) & \text{if } t_2 > t_1 \end{cases}$$

3rd ord. $\frac{1}{(i\hbar)^3}\,\frac{1}{3!}\int_0^t dt_1 \int_0^t dt_2 \int_0^t dt_3\, T\left[H(t_1)\,H(t_2)\,H(t_3)\right]$

$$U(t) = T\left(e^{-\frac{i}{\hbar}\int^t H(s)\,ds} \right)$$

A practical way of using the time-ordering exponential

$$\mathcal{U}(t) = T\left(e^{\frac{1}{i\hbar}\int^{t} H(s)ds} \right) \qquad n\Delta t = t$$

$$= T\left(e^{\frac{1}{i\hbar}\sum_{i=0}^{n} H(i\Delta t)\Delta t} \right)$$

$$= e^{\frac{1}{i\hbar}\Delta t H(n\Delta t)} e^{\frac{1}{i\hbar}\Delta t H((n-1)\Delta t)} \cdots e^{\frac{1}{i\hbar}\Delta t H(0)}$$

exact time ordering when $\Delta t \to 0$.