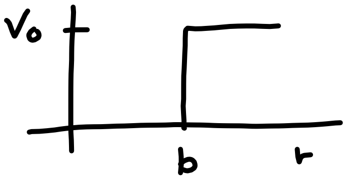
For a finite well, needs to compute $R_\ell(kb)$ & $R_\ell'(kb)$



$R_\ell = e^{i\delta}(j_\ell c_\ell - n_\ell s_\ell)$ $\quad s_\ell = \sin\delta_\ell$

$R_\ell' = e^{i\delta}(j_\ell' c_\ell - n_\ell' s_\ell)$ $\quad c_\ell = \cos\delta_\ell$

Let $\alpha_\ell = \frac{R_\ell'}{R_\ell} = \frac{\partial}{\partial r}\ln R_\ell = \frac{j_\ell' c_\ell - n_\ell' s_\ell}{j_\ell c_\ell - n_\ell s_\ell}$

$$\alpha_\ell j_\ell c_\ell - \alpha n_\ell s_\ell = j_\ell' c_\ell - n_\ell' s_\ell$$

$$s_\ell(n_\ell' - \alpha_\ell n_\ell) = c_\ell(j_\ell' - \alpha_\ell j_\ell)$$

$$\cot\delta_\ell = \frac{n_\ell' - \alpha_\ell n_\ell}{j_\ell' - \alpha_\ell j_\ell}$$

first for $V_0 \to \infty$
$\alpha \to \infty$

$$= \frac{n_\ell}{j_\ell}\left(\frac{\frac{n_\ell'}{n_\ell} - \alpha_\ell}{\frac{j_\ell'}{j_\ell} - \alpha_\ell}\right) = \frac{n_\ell}{j_\ell}\left(\frac{(\ln n_\ell)' - \alpha_\ell}{(\ln j_\ell)' - \alpha_\ell}\right)$$

For low energy scattering
$kb \ll 1$

$$j_\ell = \frac{1}{(2\ell+1)!!}(kr)^\ell, \quad n_\ell = -(2\ell-1)!!(kr)^{-(\ell+1)}$$

$$\cot\delta_\ell = +(kr)^{-(2\ell+1)}(2\ell+1)!!(2\ell-1)!!$$

at $r=b$ $\quad \left(\frac{-(\ell+1)\frac{1}{r} - \alpha_\ell}{\ell\frac{1}{r} - \alpha_\ell}\right)$

$$= \left(\frac{+(\ell+1) + \alpha_\ell b}{\ell - \alpha_\ell b}\right)$$

$(2\ell+1)!! = 1\cdot 3\cdot 5\cdots$

$\ln j_\ell = \ell \ln r$

$(\ln j_\ell)' = \frac{\ell}{r}$

$\ln n_\ell = -(\ell+1)\ln r$

$(\ln n_\ell)' = -(\ell+1)\frac{1}{r}$

for $\ell = 0$ $\quad \cot\delta_0 = \frac{1}{kb}\,\frac{1+\alpha_0 b}{-\alpha_0 b} \Rightarrow \boxed{k\cot\delta_0 = -\frac{1}{a}}$

scattering length

$$k\frac{\cos\delta_0}{\sin\delta_0} = -\frac{1}{a}$$

in the limit of $k \to 0$

$$k\frac{1}{\delta_0} = -\frac{1}{a}$$

$$\boxed{\delta_0 = -ka}$$

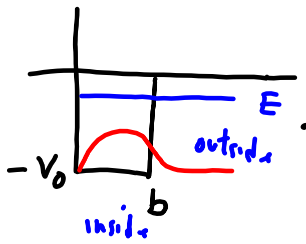
The case of a square well (3D)

Scattering $E > 0$, near $E = 0$

bound state $E < 0$, " $E = 0$

$\kappa = \frac{1}{a}$

$$E = -\frac{\hbar^2 \kappa^2}{2m} = -\frac{\hbar^2}{2m}\frac{1}{a^2}$$



for bound state $E < 0$.

inside	outside
$\sin(Kr)$	$e^{-\kappa r}$
$K\cos(Kr)$	$-\kappa e^{-\kappa r}$

$r = b$

$$E = -\frac{\hbar^2 \kappa^2}{2m}$$

$$\frac{\hbar^2 K^2}{2m} = E - V = E + V_0$$

$$K = \sqrt{\frac{2mV_0}{\hbar^2} - \kappa^2}$$

$$\boxed{bK \cot Kb = -\kappa b}$$

near $E \sim 0$, $\kappa \sim 0$ $K \to K_0 = \sqrt{\frac{2mV_0}{\hbar^2}}$

$$\Rightarrow K_0 b = \frac{\pi}{2} \checkmark \qquad \cot = \frac{\cos x}{\sin x}$$

$$g(x) = x\cot x = x\frac{\cos x}{\sin x} \quad \text{expand about } x = \frac{\pi}{2}$$

$$g(x) = g(\tfrac{1}{2}\pi) + g'(x = \tfrac{1}{2}\pi)(x - \tfrac{1}{2}\pi)$$

$$g(x) = 0 + (x - \tfrac{1}{2}\pi)(-\tfrac{\pi}{2})$$

near $E = 0$

$$\boxed{\frac{\pi}{2}\left(K_0 b - \frac{\pi}{2}\right) = b\kappa}$$

for $E > 0$

$$E = \frac{\hbar^2 k^2}{2m}$$

$$K = \sqrt{\frac{2m}{\hbar^2}V_0 + k^2}$$

Inside	outside
$\sin(Kr)$	$= \sin(kr + \delta_0)$
$K\cos(Kr)$	$= k\cos(kr + \delta_0)$

$$\frac{1}{K}\tan(Kb) = \frac{1}{k}\tan(kb + \delta_0)$$

$k \to 0$ $K \to K_0$

$$\frac{1}{K_0}\tan(K_0 b) = b + \frac{\delta_0}{k}$$

$$\delta_0 = -ka$$

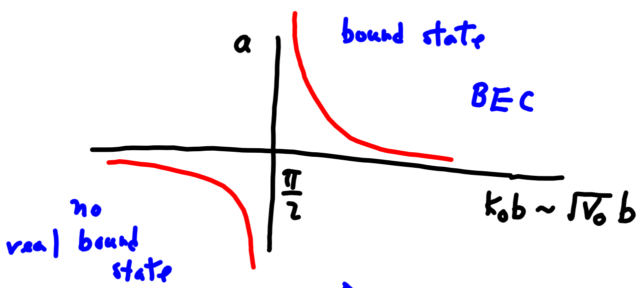
$$= b - a$$

$$a = b - \frac{b}{K_0 b}\tan(K_0 b)$$

$$= b - \frac{b}{K_0 b}\frac{1}{\cot(K_0 b)}$$

$a >> b$

$$a = +\frac{b}{\frac{\pi}{2}(K_0 b - \frac{\pi}{2})}$$



$$a = b\frac{1}{b\kappa} = \frac{1}{\kappa}$$

$$\Rightarrow E = -\frac{\hbar^2 \kappa^2}{2m} = -\frac{\hbar^2}{2m}\frac{1}{a^2}$$

Feshback resonance

For BEC at $a = \infty$ unitary limit

1) Optical theorem:

$$f(\theta) = \frac{1}{k}\sum_{\ell}(2\ell+1)\,P_\ell(\cos\theta)\,\underbrace{e^{i\delta_\ell}}_{\cos\delta_\ell + i\sin\delta_\ell}\sin\delta_\ell$$

forward scattering $\theta = 0$

$$\mathrm{Im}\, f(0) = \frac{1}{k}\sum_{\ell}(2\ell+1)\sin^2\delta_\ell = \frac{k}{4\pi}\sigma$$

total cross-section

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell}(2\ell+1)\sin^2\delta_\ell$$

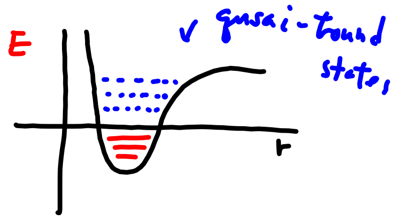
$$\sigma = \frac{4\pi}{k}\,\mathrm{Im}\, f(0)$$

interfered to give zero

depletion of the original incoming wave

"total scattering"

2) Resonance scattering



example

triple $\alpha \to C^{12}$

in stellar temperature

$\alpha + \alpha \to {}^8Be^* + \alpha \to C^{12}$

resonance scattering

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\frac{\sin^2\delta_\ell}{\cos^2\delta_\ell + \sin^2\delta_\ell} = \frac{4\pi}{k^2}(2\ell+1)\frac{1}{\cot^2\delta_\ell + 1}$$

Since

$$\cot\delta_\ell = (kb)^{-2\ell-1}(2\ell-1)!!\,(2\ell+1)!!\,\frac{\ell+1+b\alpha_\ell(E)}{\ell - b\alpha_\ell}$$

so at $k \sim 0$ $\cot \to \infty$ $\sigma_\ell \to 0$

unless $\boxed{\ell+1+b\,\alpha_\ell(E) = 0}$ resonance condition

$f(E)$

$= \ell+1+b\alpha_\ell(E)$

at $E = E_r$ ↓ condition for a bound stat

$= f(E_r) + (E-E_r)f'(E_r)$

also

$= 0 + (E-E_r)\,b\,\alpha_\ell'(E_r)$ $\boxed{\ell+1+b\,\alpha_\ell(-E_b) = 0}$

$b\alpha_\ell \approx -\ell-1$

$\Rightarrow \ell - b\alpha_\ell = 2\ell+1$

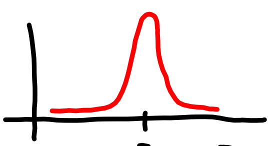
$$\cot\delta_\ell = (kb)^{-2\ell-1}[(2\ell-1)!!]^2(E-E_r)\,b\,\frac{\partial\alpha}{\partial E}$$

$$\Gamma_R = -\frac{2k^{2\ell+1}b^{2\ell}}{[(2\ell-1)!!]^2\frac{\partial\alpha}{\partial E}}$$

$$= -\frac{2}{\Gamma_R}(E-E_r) = \cot\left(x+\frac{\pi}{2}\right) = -\tan x$$

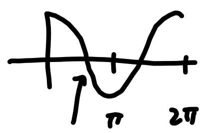
$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\frac{\Gamma^2}{4(E-E_r)^2 + \Gamma^2}$$

$x = \tan^{-1}\left(\frac{2(E-E_r)}{\Gamma_R}\right)$



$$\boxed{\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\frac{(\Gamma_R/2)^2}{(E-E_r)^2 + (\Gamma_R/2)^2}}$$

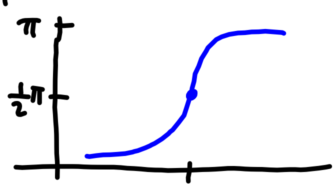
at resonance $\cot\delta_\ell = \frac{\cos\delta_\ell}{\sin\delta_\ell} = 0 \Rightarrow$ $\delta_\ell = \frac{1}{2}\pi$



Let $\delta_\ell = \frac{\pi}{2} + x$

$\cot\left(\frac{\pi}{2}+x\right) = -\tan(x)$

but



behavior of δ_ℓ near a resonance.

Loose end : Determine the phase-shift in the integral form

$$f = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)\, e^{i\delta_\ell}\sin\delta_\ell \qquad \vec{k}' = k\hat{r}$$

$$= -\frac{m}{2\pi\hbar^2}\int d^3\vec{r}'\, e^{-i\vec{k}'\cdot\vec{r}'}\, V(\vec{r}')\,\psi_k(\vec{r}')$$

compare

$$4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^{\ell}\, Y_{\ell m}(\theta,\phi)\, j_\ell(kr)\, Y^*_{\ell m}(\theta',\varphi') \qquad \sum_{\ell'=0}^{\infty} i^{\ell'}(2\ell'+1)\underbrace{P_{\ell'}(\cos\theta')}_{Y_{\ell' 0}(\theta',\varphi')\sqrt{\frac{4\pi}{2\ell'+1}}}R_{\ell}(kr)$$

$\delta_{\ell\ell'}\ \delta_{m0}$

$$\boxed{f = -\frac{m}{\hbar^2}\, 2\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)\int_0^{\infty} dr\, r^2\, V(r)\, j_\ell(kr)\, R_\ell(kr)}$$

$$\frac{1}{k}e^{i\delta_\ell}\sin\delta_\ell = -\frac{2m}{\hbar^2}\int_0^{\infty} r^2 dr\, V(r)\, j_\ell(kr)\, R_\ell(kr)$$

For the Born approximation $R_\ell(kr) \sim j_\ell(kr)$

$$e^{i\delta_\ell}\sin\delta_\ell = -\frac{2mk}{\hbar^2}\int_0^{\infty} r^2 dr\, V(r)\, j_\ell^2(kr)$$

$\simeq \delta_\ell^B =$