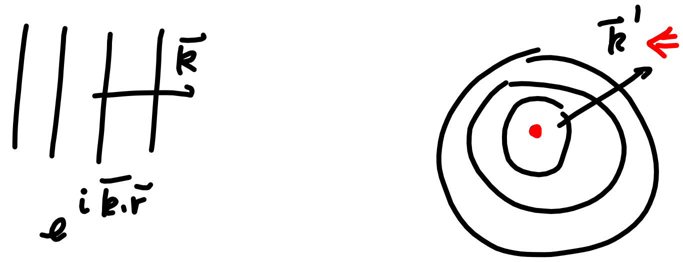


$$\psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\int d^3\vec{r}\,' \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} V(\vec{r}\,')\psi_k(\vec{r}\,')$$

$$\vec{r} \gg \vec{r}\,' \qquad |\vec{r}-\vec{r}\,'| = \sqrt{r^2 - 2\vec{r}\cdot\vec{r}\,' + \dots} \approx r\sqrt{1 - 2\frac{\vec{r}}{r^2}\cdot\vec{r}\,'}$$

$$\approx r\left(1 - \frac{\vec{r}}{r^2}\cdot\vec{r}\,' \dots\right) = r - \hat{r}\cdot\vec{r}\,' + \dots$$

$$k|\vec{r}-\vec{r}\,'| \to kr - \underbrace{(k\hat{r})}_{k'}\cdot\vec{r}\,'$$

$$\psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \underbrace{\left[-\frac{m}{2\pi\hbar^2}\int d^3\vec{r}\,' e^{-i\vec{k}'\cdot\vec{r}\,'} V(\vec{r}\,')\psi_k(\vec{r}\,')\right]}_{f_k(\hat{r}) = f_k(\Omega_r)} \frac{e^{ikr}}{r}$$

$f_k(\hat{r}) = f_k(\Omega_r)$ = scattering amplitude, $\hat{r} \leftarrow (\theta, \phi)$

Cross-Section :

$$d\sigma = \frac{\#\text{ particle scattered into } d\Omega \text{ per unit time}}{\#\text{ incoming particle per unit area per unit time}}$$

$$= \frac{\frac{|f|^2}{r^2}\left(\frac{\hbar k}{m}\right) r^2 d\Omega}{1\left(\frac{\hbar k}{m}\right)} \Rightarrow \frac{d\sigma}{d\Omega} = |f|^2$$

Born scattering ~ V is weak or $k \gg 1$

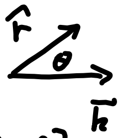
$$\psi_k(\vec{r}\,') \sim e^{i\vec{k}\cdot\vec{r}\,'}$$

$$f_k = -\frac{m}{2\pi\hbar^2}\int d^3r'\, e^{-i\vec{k}'\cdot\vec{r}\,'}\, V(r')\, e^{i\vec{k}\cdot\vec{r}\,'}$$

$$= -\frac{m}{2\pi\hbar^2}\langle \vec{k}'|V|\vec{k}\rangle \qquad |\vec{k}'| = |\vec{k}|$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\langle \vec{k}'|V|\vec{k}\rangle\right|^2 \qquad \text{elastic scattering}$$

need to know: $(\vec{k}-\vec{k}')^2 = k^2 + k^2 - 2\vec{k}\cdot\vec{k}'$



$$= 2\cdot 2k^2\left(\frac{1-\cos\theta}{2}\right) = \left[2k\sin\frac{\theta}{2}\right]^2$$

For a Yukawa potential

$$V(r) = V_0\frac{e^{-\alpha r}}{r}$$

nuclear, shielded - Coulomb potential

$$\langle \vec{k}'|V|\vec{k}\rangle = \int d^3\vec{r}\,'\, V(r')\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}\,'}$$

Recall: $$G(r,k) = -\frac{\cancel{m}}{\cancel{\hbar^2}}\frac{1}{2\pi}\frac{e^{ikr}}{r} = \frac{2\cancel{m}}{\cancel{\hbar^2}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{k^2-q^2}e^{i\vec{q}\cdot\vec{r}}$$

$k \to i\alpha$ $$\cancel{-}\frac{1}{2\cancel{\pi}}\frac{e^{-\alpha r}}{r} = \cancel{-}2\int\frac{d^3q}{(2\pi)^3}\frac{1}{\alpha^2+q^2}e^{i\vec{q}\cdot\vec{r}}$$

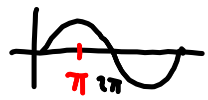
$$\frac{e^{-\alpha r}}{r} = 4\pi\int\frac{d^3q}{(2\pi)^3}\frac{1}{\alpha^2+q^2}e^{i\vec{q}\cdot\vec{r}} \qquad \Delta k = \vec{k}'-\vec{k}$$

$$\langle \vec{k}'|V|\vec{k}\rangle = \frac{4\pi}{\cancel{(2\pi)^3}}\int d^3q\,\frac{1}{\alpha^2+q^2}\underbrace{\int d^3\vec{r}\,'\, e^{i(\vec{q}-\Delta\vec{k})\cdot\vec{r}\,'}}_{\cancel{(2\pi)^3}\,\delta(\vec{q}-\Delta\vec{k})}$$

$$= 4\pi\frac{V_0}{\alpha^2+(\Delta\vec{k})^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{\cancel{4\pi^2}\hbar^4}\frac{\cancel{4\pi}\,\cancel{4\pi}\,V_0^2}{\left[\alpha^2+\left[4k^2\sin^2\frac{\theta}{2}\right]\right]^2}$$

$\sin\frac{\theta}{2}$



$$\frac{d\sigma}{d\Omega} = \frac{V_0^2}{\left[\frac{\hbar^2\alpha^2}{2m} + 4E_k\sin^2\frac{\theta}{2}\right]^2} \xrightarrow[\alpha\to 0]{} \frac{V_0^2}{\left[4E_k\sin^2\frac{\theta}{2}\right]^2}$$

Coulomb scattering

High energy scattering → Born

Low " " → partial waves

For spherical symmetric potential ⟷ angular momentum is conserved

⇒ Scattering wave are eigenstate of $\vec{L}$

⇒ The ℓth wave of the incoming wave → ℓth partial wave of the outgoing wave.

Plane wave decomposition:

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) P_\ell(\cos\theta) \underbrace{J_\ell(kr)}_{\frac{1}{2}\left(h_\ell(kr) + h_\ell^*(kr)\right)}$$

For the exact solution

$$\psi_k(\vec{r}) = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) P_\ell(\cos\theta) R_\ell(kr)$$

$$\left(\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2}\right) R_\ell = \frac{2m}{\hbar^2} V(r)\, r R_\ell$$

at $r \to \infty$

$$R_\ell(r) \to \alpha h_\ell(kr) + \beta h_\ell^*(kr)$$

$$\to B_\ell\left[h_\ell^*(kr) + S_\ell(\epsilon) h_\ell(kr)\right]$$

$$\to \frac{1}{2}\left[h_\ell^*(kr) + S_\ell(\epsilon) h_\ell(kr)\right]$$

$h_\ell \sim \frac{e^{ikr}}{r}$

unitarity → con. of prob. ⇒ $|S_\ell(\epsilon)|^2 = 1$

$$\boxed{S_\ell(\epsilon) = e^{2i\delta_\ell}}$$

δ_ℓ = phase shift

$$\psi_k(r) = \frac{1}{2}\sum_\ell i^\ell (2\ell+1) P_\ell(\cos\theta)\left[h_\ell^*(kr) + e^{2i\delta_\ell} h_\ell(kr)\right]$$

$r \to \infty$ $\qquad + h_\ell(kr) - h_\ell(kr)$

$$= e^{i\bar{k}\cdot\bar{r}} + \frac{1}{2}\sum_\ell i^\ell (2\ell+1) P_\ell(\cos\theta)\left[e^{i2\delta_\ell} - 1\right] h_\ell(kr)$$

$$h_\ell = \frac{e^{i(kr - \frac{\ell\pi}{2}) - i\frac{\pi}{2}}}{kr}$$

$$= e^{i\bar{k}\cdot\bar{r}} + \frac{1}{2}\sum_\ell i^\ell (2\ell+1) P_\ell \left[e^{i2\delta_\ell} - 1\right]\frac{e^{ikr}}{kri}$$

$(-i)^\ell$

$$f = \frac{1}{2ki}\sum_\ell (2\ell+1) P_\ell(\cos\theta)\left[e^{i2\delta_\ell} - 1\right]$$

$e^{i\delta_\ell}\left[e^{i\delta_\ell} - e^{-i\delta_\ell}\right]$

$$\frac{d\sigma}{d\Omega} = \boxed{f = \frac{1}{k}\sum_\ell (2\ell+1) P_\ell(\cos\theta)\, e^{i\delta_\ell}\sin\delta_\ell}$$

$$\sigma = \int |f|^2 d\Omega = \int d\Omega \frac{1}{k^2}\sum_\ell (2\ell+1) P_\ell(\cos\theta)\, e^{i\delta_\ell}\sin\delta_\ell \sum_{\ell'} (2\ell'+1) P_{\ell'}(\cos\theta)\, e^{-i\delta_{\ell'}}\sin\delta_{\ell'}$$

$L^2 = \ell(\ell+1)$

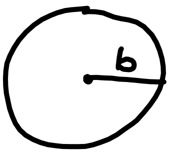
$\sim (\ell+\frac{1}{2})^2$

$L = (\ell+\frac{1}{2})\hbar$

$$\int d\Omega\, P_\ell P_{\ell'} = 4\pi \frac{\delta_{\ell\ell'}}{2\ell+1}$$

$\sin\theta\, d\theta\, d\phi$

$$\boxed{\sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell}$$



$$\sigma \leqslant \frac{4\pi}{k^2}\sum_\ell (2\ell+1) \qquad \sin^2\delta_\ell \leqslant 1$$

$L = bp = b\hbar k$

$(\ell+\frac{1}{2})\hbar = b\hbar k$ (max)

$$\leqslant \frac{4\pi}{k^2}\, 2\sum_{\ell}^{bk} \underbrace{(\ell+\tfrac{1}{2})}_{n} \to \frac{1}{2}(bk)(bk+) \sim \frac{1}{2}b^2k^2$$

$\frac{1}{2} = 1$

$$\leqslant 4\pi b^2 \quad \text{upp bound}$$

$\langle \sin^2\delta_\ell \rangle \simeq \frac{1}{2}$

$$\sigma \sim 2\pi b^2 \longrightarrow f = f_{scatt} + f_{shadow}$$

$$|f|^2 \sim \pi b^2 + \pi b^2$$

The meaning of δ_ℓ

Asymptotically: $R_\ell \sim \frac{1}{2} e^{i\delta_\ell}\left[h_\ell^* e^{-i\delta_\ell} + h_\ell e^{i\delta_\ell}\right]$

$r \to \infty$

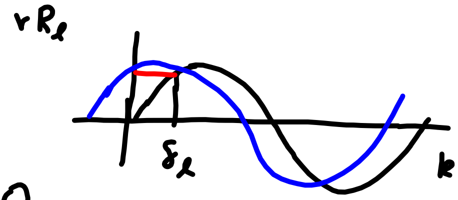
$$\sim e^{i\delta_\ell} \frac{1}{kr}\left[e^{-ikr-i\delta_\ell} + e^{+ikr+i\delta_\ell}\right] e^{-i(\ell+1)\frac{\pi}{2}}$$

$$\sim e^{i\delta_\ell} \frac{1}{kr} \cos\left(kr + \delta_\ell - (\ell+1)\frac{\pi}{2}\right)$$

$$e^{i\delta_\ell} \frac{1}{kr} \sin\left(kr + \delta_\ell - \ell\frac{\pi}{2}\right)$$

no scattering $\sin(kr)$

with scattering $\sin(kr+\delta_\ell)$

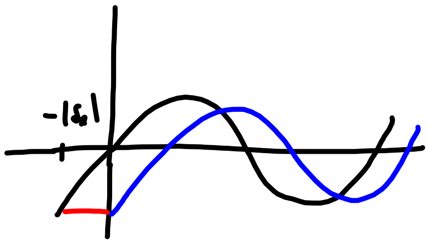


if $\delta_\ell > 0$

the wf is shifted backward (into the pot)

↪ attractive potential

$\sin(kr-|\delta_\ell|)$



if $\delta_\ell < 0$

wf pushed out

↪ repulsive potential

Example : hard sphere scattering
(infinite potential) $E = \frac{\hbar^2 k}{2m}$

$$R_\ell(kr) = \frac{1}{2}\left(h_\ell^*(kr) + e^{i2\delta_\ell} h_\ell(kr)\right)$$

$$= \frac{1}{2} e^{i\delta_\ell}\left(h_\ell^* e^{-i\delta_\ell} + e^{i\delta_\ell} h_\ell(kr)\right)$$

$$R_\ell = e^{i\delta_\ell}\left(j_\ell \cos\delta_\ell - \eta_\ell \sin\delta_\ell\right)$$

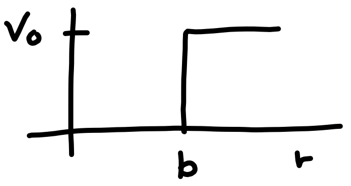
for an infinite pot
at $r=b$ $\quad j_\ell(kb)\cos\delta_\ell = \eta_\ell(kb)\sin\delta_\ell$

$$\Rightarrow \tan\delta_\ell = \frac{j_\ell(kb)}{\eta_\ell(kb)} .$$

for $\ell=0$

$$= -\frac{\sin(kb)}{\cos(kb)} \Rightarrow \boxed{\delta_0 = -kb}$$

For a finite well, needs to compute $R_\ell(kb)$ & $R_\ell'(kb)$



$R_\ell = e^{i\delta}(j_\ell c_\ell - n_\ell s_\ell)$ $\quad s_\ell = \sin\delta_\ell$

$R_\ell' = e^{i\delta}(j_\ell' c_\ell - n_\ell' s_\ell)$ $\quad c_\ell = \cos\delta_\ell$

Let $\alpha_\ell = \frac{R_\ell'}{R_\ell} = \frac{\partial}{\partial r}\ln R_\ell = \frac{j_\ell' c_\ell - n_\ell' s_\ell}{j_\ell c_\ell - n_\ell s_\ell}$

$$\alpha_\ell j_\ell c_\ell - \alpha n_\ell s_\ell = j_\ell' c_\ell - n_\ell' s_\ell$$

$$s_\ell(n_\ell' - \alpha_\ell n_\ell) = c_\ell(j_\ell' - \alpha_\ell j_\ell)$$

$$\cot\delta_\ell = \frac{n_\ell' - \alpha_\ell n_\ell}{j_\ell' - \alpha_\ell j_\ell}$$

first for $V_0 \to \infty$, $\alpha \to \infty$

$$= \frac{n_\ell}{j_\ell}\left(\frac{\frac{n_\ell'}{n_\ell} - \alpha_\ell}{\frac{j_\ell'}{j_\ell} - \alpha_\ell}\right) = \frac{n_\ell}{j_\ell}\left(\frac{(\ln n_\ell)' - \alpha_\ell}{(\ln j_\ell)' - \alpha_\ell}\right)$$

For low energy scattering $kb \ll 1$

$$j_\ell = \frac{1}{(2\ell+1)!!}(kr)^\ell, \quad n_\ell = -(2\ell-1)!!(kr)^{-(\ell+1)}$$

$$\cot\delta_\ell = +(kr)^{-(2\ell+1)}(2\ell+1)!!(2\ell-1)!!\left(\frac{-(\ell+1)\frac{1}{r} - \alpha_\ell}{\ell\frac{1}{r} - \alpha_\ell}\right)$$

at $r=b$

$(2\ell+1)!! = 1\cdot 3\cdot 5\cdots$

$\ln j_\ell = \ell \ln r$

$(\ln j_\ell)' = \frac{\ell}{r}$

$\ln n_\ell = -(\ell+1)\ln r$

$(\ln n_\ell)' = -(\ell+1)\frac{1}{r}$

$$'' \left(\frac{+(\ell+1)+\alpha_\ell b}{\ell - \alpha_\ell b}\right)$$

for $\ell = 0$ $\quad \cot\delta_0 = \frac{1}{kb}\frac{1+\alpha_0 b}{-\alpha_0 b} \Rightarrow$ $\boxed{k\cot\delta_0 = -\frac{1}{a}}$

a: scattering length

in the limit of $k \to 0$