Project 2 will due Next Thursday
Feb. 1, 5 pm, at my mailbox

Offic hour M + T → T + W

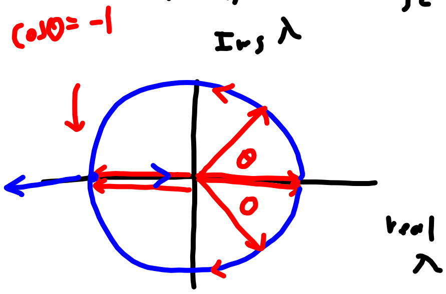
No class on Monday → Tuesday 7 pm

Last time : stability of an algorithm

1) requires $\boxed{\det M = 1}$

2) the range of stability is given $\boxed{\left|\frac{T}{2}\right| < 1}$

Recall $\lambda_{1,2} = \frac{T}{2} \pm i\sqrt{1-\left(\frac{T}{2}\right)^2}$ if $\det M = 1$




$= \cos\theta \pm i \sin\theta$ $\quad \cos\theta = \frac{T}{2}$

$= e^{\pm i\theta}$ $\quad |\lambda_{1,2}| = 1$

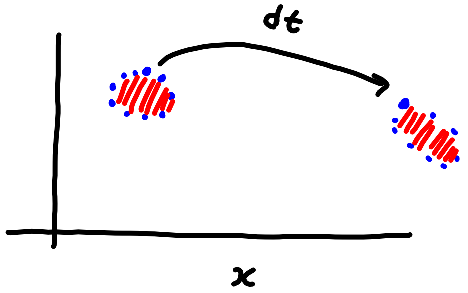
at $\Delta t = 0$ $\quad M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \theta = 0$

## The meaning of det M = 1

1) Liouville's Theorem : fundamental property of classical mechanics.

$$dx_{n+1}\, dv_{n+1} = dx_n\, dv_n$$

$v = \frac{p}{m}$




area enclosed is unchange.

preservation of phase - space

$x' = x_{n+1}$

$x = x_n$

2) $$\det \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial v} \\ \frac{\partial v'}{\partial x} & \frac{\partial v'}{\partial v} \end{pmatrix} = \frac{\partial x'}{\partial x}\frac{\partial v'}{\partial v} - \frac{\partial x'}{\partial v}\frac{\partial v'}{\partial x}$$

$v = \frac{p}{m}$

$$= \{x', v'\}_{x,v} = 1$$ Poisson Bracket !

$\{x', x'\}_{x v} = 0$ $\{v', v'\}_{x v} = 0$

} fundamental Poisson bracket

$\det M = 1$ $\Rightarrow (x, v) \rightarrow (x', v')$ $\Leftarrow$

is a canonical transformation

symplectic map

Well known algorithms → Runge-Kutta

↑ non-symplectic.

↑ Taylor expansion.

Solve second-order differential equation based on Newton's equation

$$\frac{d^2\vec{r}}{dt^2} = \vec{a}(\vec{r}) \qquad \vec{v} = \frac{d\vec{r}}{dt}$$

$$\frac{d}{dt}\begin{pmatrix}\vec{r}\\ \vec{v}\end{pmatrix} = \begin{pmatrix}\vec{v}\\ \vec{a}\end{pmatrix} \Rightarrow \frac{d}{dt}\bar{X} = \bar{V}(\bar{X})$$

$\bar{X}$ " $V$

$$\bar{X}(\Delta t) = \bar{X}_0 + \Delta t\,\dot{\bar{X}} + \frac{1}{2}\Delta t^2\,\ddot{\bar{X}}$$ ← keep second order term

$$= \bar{X}_0 + \Delta t\left(V + \frac{1}{2}\Delta t\,\dot{V}\right)$$

$$= \bar{X}_0 + \Delta t\left(V\left(\frac{1}{2}\Delta t\right)\right)$$

$$\begin{pmatrix}\vec{r}_1\\ \vec{v}_1\end{pmatrix} = \begin{pmatrix}\vec{r}_0\\ \vec{v}_0\end{pmatrix} + \Delta t\begin{pmatrix}\vec{v}_0 + \frac{1}{2}\Delta t\,\vec{a}(\vec{r}_0)\\ \vec{a}\left(\vec{r}_0 + \frac{1}{2}\Delta t\,v_0\right)\end{pmatrix}$$

RK2

$$\boxed{\begin{aligned}\vec{r}_{n+1} &= \vec{r}_n + \Delta t\,\vec{v}_n + \frac{1}{2}\Delta t^2\,\vec{a}_n\\ \vec{v}_{n+1} &= \vec{v}_n + \Delta t\,\vec{a}\left(\vec{r}_n + \frac{1}{2}\Delta t\,\vec{v}_n\right)\end{aligned}}$$

$\vec{a}_n = a(\vec{r}_n)$

$\det M \neq 1$

4$^{\underline{th}}$ order Runge-Kutta-Nyström

only requires 3-force evaluations for $\vec{a}(\vec{r})$ not dependent on $t$.

$$\vec{r}_{n+1} = \vec{r}_n + \Delta t\,\vec{v}_n + \frac{\Delta t^2}{6}\left(\vec{a}_n + 2\vec{a}_{n1}\right)$$

$$\vec{v}_{n+1} = \vec{v}_n + \frac{\Delta t}{6}\left(\vec{a}_n + 4\vec{a}_{n1} + \vec{a}_{n2}\right)$$

$\vec{a}_n = \vec{a}(\vec{r}_n)$ $\quad$ $\vec{r}_n$ initial

$\vec{a}_{n1} = \vec{a}(\vec{r}_{n1})$ $\quad$ middle $\vec{r}_{n1} = \vec{r}_n + \frac{1}{2}\Delta t\,\vec{v}_n + \frac{1}{2}\left(\frac{1}{2}\Delta t\right)^2 \vec{a}_n$

$\vec{a}_{n2} = \vec{a}(\vec{r}_{n2})$ $\quad$ $\vec{r}_{n2} = \vec{r}_n + \Delta t\,\vec{v}_n + \frac{1}{2}\Delta t^2\,\vec{a}_{n1}$

final.