Last time : 1) The errors of an algorithm propagate according to the Jacobian matrix M of the transformation $(x_n, v_n) \to (x_{n+1}, v_{n+1})$

2) stability of an algorithm requires $|\lambda_i| = 1$!

For 2x2 matrix the condition for determining the eigenvalue is

$$\det\left(M - \lambda I\right) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$T =$ trace of M $\quad ad - \lambda(a+d) + \lambda^2 - bc = 0$

$D = \det M$ $\quad \lambda^2 - (a+d)\lambda + (ad-bc) = 0$

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2} = \frac{T}{2} \pm \sqrt{\left(\frac{T}{2}\right)^2 - D}$$

Note that $\quad T = \lambda_1 + \lambda_2 \quad D = \lambda_1 \lambda_2$

If $\lambda_{1,2}$ are complex $\lambda_2 = \lambda_1^*$

$$D = \lambda_1 \lambda_1^* = |\lambda_{1,2}|^2 \Rightarrow$$

if $|\lambda_{1,2}| = 1$ then

stability requires $\Rightarrow$ $\boxed{D = 1}$

when $D=1$, then

$$\lambda_{1,2} = \frac{T}{2} \pm i\sqrt{1-\left(\frac{T}{2}\right)^2}$$

Complex when $\boxed{\left|\frac{T}{2}\right| < 1}$, define θ such that $\cos\theta = \frac{T}{2}$

stable

$$\lambda_{1,2} = \cos\theta \pm i\sin\theta$$

$$= e^{\pm i\theta}$$ complex with unit modulus

for stability

$\Rightarrow$ 1) $\mathrm{Det}\, M = 1$ general

2) $|\mathrm{Tr}\, M| < 2$ ← determine the range of Δt for stability.

Compare Euler + Cromer algorithm

for $a(x) = -\frac{k}{m} x$ $\quad m \frac{d^2x}{dt^2} = -kx$

$= -\omega^2 x$

For Euler : $x_{n+1} = x_n + \Delta t\, v_n$; $v_{n+1} = v_n + \Delta t\, a(x_n)$

$= v_n - \Delta t\, \omega^2 x_n$

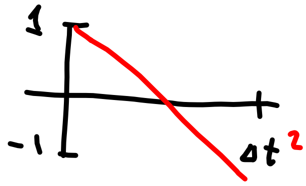
$$M = \begin{pmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial v_n} \\ \frac{\partial v_{n+1}}{\partial x_n} & \frac{\partial v_{n+1}}{\partial v_n} \end{pmatrix} = \begin{pmatrix} 1 & \Delta t \\ -\Delta t\, \omega^2 & 1 \end{pmatrix}$$

$$\det M = 1 + \Delta t^2 \omega^2 > 1 \qquad |\lambda_{1,2}| = \sqrt{\det M} > 1$$

For Cromer : $v_{n+1} = v_n - \Delta t\, \omega^2 x_n$; $x_{n+1} = x_n + \Delta t\, v_{n+1}$

$$\det M \overset{\det}{=} \begin{pmatrix} 1 - \Delta t^2 \omega^2 & \Delta t \\ -\Delta t\, \omega^2 & 1 \end{pmatrix} = 1 \ !$$

more over $\quad \frac{T}{2} = 1 - \frac{\Delta t^2 \omega^2}{2} > -1$



$\left|\frac{T}{2}\right| < 1 \qquad 2 > \frac{\Delta t^2 \omega^2}{2}$

$\frac{4}{\omega^2} > \Delta t^2 \qquad \frac{2}{\omega} > |\Delta t|$

How can one derive algorithms

with $\det M = 1$?

can't be derived from Taylor expansion.

Hint: $\det M = 1$

means that $\underbrace{dx_n\, dv_n}_{\text{phase-space}} = \underset{\overset{\|}{1}}{\det M}\, dx_{n+1}\, dv_{n+1}$

Liouville's Thm $\Longrightarrow$ phase space is preserved by classical (Hamiltonian) mechanics

numerical stability $\Longleftrightarrow$ phase space preservation

canonical transformations

respect fundamental aspects of physics!