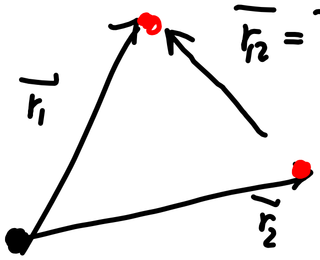
Problem : Keplerian orbit



$\bar r_{12} = \bar r_1 - \bar r_2$

$$\cancel{m_1}\frac{d^2\bar r_1}{dt^2} = -G\cancel{m_1}m_2\frac{\bar r_{12}}{|\bar r_{12}|^3}$$

$$\cancel{m_2}\frac{d^2\bar r_2}{dt^2} = Gm_1\cancel{m_2}\frac{\bar r_{12}}{|\bar r_{12}|^3}$$

Most important:

use dimensionless equation!

$$\frac{d^2\bar r_{12}}{dt^2} = -G(m_1+m_2)\frac{\bar r_{12}}{|\bar r_{12}|^3}$$

$r = r^* a$ ← unit to be determine

$t = t^* \tau$

$$\frac{\cancel{a}}{\tau^2}\frac{d^2 r_{12}^*}{dt^{*2}} = -G(m_1+m_2)\frac{\vec r_{12}^{\,*}\cancel{a}}{|r_{12}^*|^3 a^3}$$

$$\frac{d^2 r_{12}^*}{dt^{*2}} = \underbrace{-\frac{\tau^2}{a^3}G(m_1+m_2)}_{1}\frac{\vec r_{12}^{\,*}}{|r_{12}^*|^3}$$

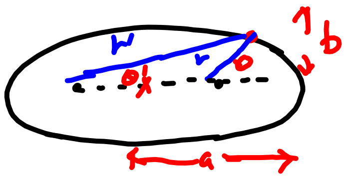
choose τ + a such that

$$\frac{d^2\bar{r}}{dt^2} = -\frac{\bar{r}}{|\bar{r}|^3}$$

r = relative vector
$= r_{12}^*$

Trajectory of the Kepler orbit:

$$r = \frac{\mathcal{P}}{1+e\cos\theta}$$



a = semi-major axis

e = eccentricity

$$r' = \frac{\mathcal{P}}{1-e\cos\theta'}$$

$$\mathcal{P} = a(1-e^2)$$

$$e = \sqrt{1-\mathcal{P}/a} = \sqrt{1-b^2/a^2}$$

determined by a + e. ← in turn, determined by the physical condition

→ geometry

physics:

1) $E = \frac{1}{2}v^2 - \frac{1}{r} = -\frac{1}{2a}$ energy determines a

2) $\mathcal{P} = h^2$ $\quad \vec{h} = \vec{r}\times\vec{v}\ (= L/m)$ angular momentum/mass determines $\mathcal{P}$ or e.

Orbit Computation: $\frac{d^2\vec{r}}{dt^2} = \vec{a}(\vec{r}) = -\frac{\vec{r}}{|\vec{r}|^2}$

$\vec{a}(\vec{r}) = \vec{F}(r)/m$

1) Taylor expansion: given $\vec{r}_0, \vec{v}_0$ at $t=0$

$\dot{\vec{r}} = \frac{d\vec{r}}{dt}$

$\ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2}$

$$\vec{r}(\Delta t) = \vec{r}(0) + \Delta t\,\dot{\vec{r}} + \frac{1}{2}\Delta t^2\,\ddot{\vec{r}} + \cdots$$

$$= \vec{r}_0 + \Delta t\,\vec{v}_0 + \frac{1}{2}\Delta t^2\,\vec{a}(\vec{r}_0) + \cdots$$

$$\vec{v}(\Delta t) = \vec{v}_0 + \Delta t\,\dot{\vec{v}} + \frac{1}{2}\Delta t^2\,\ddot{\vec{v}} + \cdots$$

$$= \vec{v}_0 + \Delta t\,\vec{a}(\vec{r}_0) + \cdots$$

Euler algorithm: keeps two terms

$t = n\Delta t$

$$\vec{r}_{n+1} = \vec{r}_n + \Delta t\,\vec{v}_n$$

$$\vec{v}_{n+1} = \vec{v}_n + \Delta t\,\vec{a}(\vec{r}_n)$$

$\vec{a}(\vec{r}_n) = \vec{a}_n$

unstable for all Δt !

(except when $\vec{a} = \text{const}$)

Cromer algorithm:

$$\vec{v}_{n+1} = \vec{v}_n + \Delta t\,\vec{a}_n$$

$$\vec{r}_{n+1} = \vec{r}_n + \Delta t\,\vec{v}_{n+1}$$

stable, for sufficiently small Δt.

canonical transformation

Stability $\longleftrightarrow$ error propagation

1D case:

$$x_{n+1} = f(x_n, v_n, \Delta t), \quad v_{n+1} = g(x_n, v_n, \Delta t) \quad \Big\} \text{ algorithm or mapping}$$

$(x_n, v_n) \to (x_{n+1}, v_{n+1})$

$x(n\Delta t)$ ←

$$x_n = \tilde{x}_n + \epsilon_n$$

↑ exact value of $x(n\Delta t)$

$$v_n = \tilde{v}_n + \delta_n$$

when $\Delta t \ll 1$, ϵ_n, δ_n are small

if f were exact:

$$x_{n+1} = \tilde{x}_{n+1} + \epsilon_{n+1} = f(\tilde{x}_n + \epsilon_n, \tilde{v}_n + \delta_n)$$

$$= f(\tilde{x}_n, \tilde{v}_n) + \epsilon_n \frac{\partial f}{\partial x_n} + \delta_n \frac{\partial f}{\partial v_n} + \cdots$$

$\tilde{x}_{n+1} = f(\tilde{x}_n, \tilde{v}_n)$

if not $\tilde{x}_{n+1} - f(\tilde{x}_n, \tilde{v}_n)$ ← forget for now

= truncation error.

$$\epsilon_{n+1} = \frac{\partial f}{\partial x_n}\epsilon_n + \frac{\partial f}{\partial v_n}\delta_n$$

$$\delta_{n+1} = \frac{\partial g}{\partial x_n}\epsilon_n + \frac{\partial g}{\partial v_n}\delta_n$$

Error propagator matrix

$$\begin{pmatrix} \epsilon_{n+1} \\ \delta_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial v_n} \\ \frac{\partial v_{n+1}}{\partial x_n} & \frac{\partial v_{n+1}}{\partial v_n} \end{pmatrix} \begin{pmatrix} \epsilon_n \\ \delta_n \end{pmatrix}$$

M " Jacobian matrix of transformation

$(x_n, v_n) \to (x_{n+1}, v_{n+1})$

$$\boxed{dv_{n+1}\, dx_{n+1} = \det M \, dv_n \, dx_n}$$

Example $(r, \theta) \to (x, y)$ $x = r\cos\theta$ $y = r\sin\theta$

$$M = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\det M \, dr\, d\theta = r\, dr\, d\theta = dx\, dy$$

Consider the simpler case of HO

$$m\frac{d^2x}{dt^2} = -kx \qquad a(x) = \frac{F}{m} = -\frac{k}{m}x$$

$M =$ const. 2×2 matrix

$$\begin{pmatrix} \epsilon_{n+1} \\ \delta_{n+1} \end{pmatrix} = M^n \begin{pmatrix} \epsilon_1 \\ \delta_1 \end{pmatrix}$$

Let λ_1, λ_2 be the eigenvalues of M

$\begin{pmatrix} \epsilon^*_{1,2} \\ \delta^*_{1,2} \end{pmatrix}$ be the eigenvectors of M

$$M\begin{pmatrix} \epsilon^*_{1,2} \\ \delta^*_{1,2} \end{pmatrix} = \lambda_{1,2}\begin{pmatrix} \epsilon^*_{1,2} \\ \delta^*_{1,2} \end{pmatrix}$$

Let $\begin{pmatrix} \epsilon_1 \\ \delta_1 \end{pmatrix} = c_1\begin{pmatrix} \epsilon^*_1 \\ \delta^*_1 \end{pmatrix} + c_2\begin{pmatrix} \epsilon^*_2 \\ \delta^*_2 \end{pmatrix}$

$$\begin{pmatrix} \epsilon_{n+1} \\ \delta_{n+1} \end{pmatrix} = M^n\begin{pmatrix} \epsilon_1 \\ \delta_1 \end{pmatrix} = c_1\lambda_1^n\begin{pmatrix} \epsilon^*_1 \\ \delta^*_1 \end{pmatrix} + c_2\lambda_2^n\begin{pmatrix} \epsilon^*_2 \\ \delta^*_2 \end{pmatrix}$$

Errors grows exponentially according to the eigenvales λ_1, λ_2 of M.

stable algorithm requires that $|\lambda_{1,2}| = 1$.

unstable algorithm has $|\lambda_{1,2}| > 1$.