Print Name: ________________________________

PHYSICS 606 : FALL SEMESTER 2018

Homework #5

1. For the one-dimensional harmonic oscillator, let

$$\langle 0|x^2|0\rangle = v \equiv \frac{\hbar}{2m\omega},$$

   (a) use normal ordering and Wick's theorem, without doing any integral, prove that

$$\langle 0|e^{ikx}|0\rangle = e^{-k^2v/2}. \tag{1}$$

   (b) If A and B are two operators such that $[A, B]$ is just a number and not an operator, then one has the exact result

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}.$$

   Use this result and prove (1) directly without invoking Wick's theorem.

2. (Baym 5.1)

   (a) Derive and solve the equations of motion for the Heisenberg operators $a(t)$ and $a^\dagger(t)$ for the harmonic oscillator.

   (b) Calculate $[a(t), a^\dagger(t')]$.

3. (Baym 5.2) Use the result $[\mathbf{r}, f(\mathbf{p})] = i\hbar\nabla_p f(\mathbf{p})$ where $f(\mathbf{p})$ is an arbitrary function of the momentum operator and result show that

   (a)

$$e^{i\mathbf{p}\cdot\mathbf{\Lambda}/\mathbf{\hbar}}\mathbf{r}e^{-i\mathbf{p}\cdot\mathbf{\Lambda}/\mathbf{\hbar}} = \mathbf{r} + \Lambda$$

   where Λ is a numerical vector. (Hint: use the identity given in class:

$$e^{\epsilon H}Ae^{-\epsilon H} = A + \epsilon[H, A] + \frac{1}{2}\epsilon^2[H, [H, A]] + +\frac{1}{3!}\epsilon^2[H, [H, [H, A]]] + \cdots)$$

   (b) Show that the wave function of the state

$$|\Phi\rangle \equiv e^{-i\mathbf{p}\cdot\mathbf{\Lambda}/\mathbf{\hbar}}|\Psi\rangle$$

   is the same as the wave function of the state $|\Psi\rangle$, only shifted a distance Λ. Write out $\langle\mathbf{x}|\Phi\rangle$ explicitly if $|\Psi\rangle$ is the ground state of the harmonic oscillator, in one dimension.

   (c) Show that $|\Phi\rangle$ develops in the Schrödinger representation by

$$|\Phi(t)\rangle = e^{-i\mathbf{p}(-\mathbf{t})\cdot\mathbf{\Lambda}/\mathbf{\hbar}}|\Psi(t)\rangle$$

   Where $\mathbf{p}(-t)$ is the momentum operator in the Heisenberg representation at $-t$.

4. (Baym 5.4)

   (a) Calculate the correlation function $\langle 0|x(t)x(t')|0\rangle$ where $|0\rangle$ is the ground state of a one-dimensional harmonic oscillator, and x(t) is the position operator in the Heinsenberg representation.

   (b) Suppose that a time-dependent force F(t) is applied to the particle in the oscillator potential. Show that x(t) obeys the equation of motion

$$m\left(\frac{d^2}{dt^2}+w^2\right)x(t) = F(t)$$

   where w is the frequency of the oscillator.

5. (Baym 5.6)

   (a) Solve the Heisenberg equations of motion for the position and momentum of a free particle perturbed by a uniform force F(t). What is the energy and momentum transferred to the particle by the force after it acts for time t?

   (b) If at time 0 the wave function of the particle is of the form

$$\psi(t) = e^{i\mathbf{k}\cdot\mathbf{r}}\varphi(r)$$

   where $\varphi(r)$ is real, show that the uncertainty, $(\Delta r)_t$, in the position of the particle at the later t is given by

$$(\Delta r)_t = \left[(\Delta r)_0^2 + \frac{2t^2}{m}\left(\langle E\rangle - \frac{\hbar^2 k^2}{2m}\right)\right]^{1/2}$$

   regardless of the particular force applied, where $\langle E\rangle$ is the expectation value of the energy of the particle at $t = 0$.