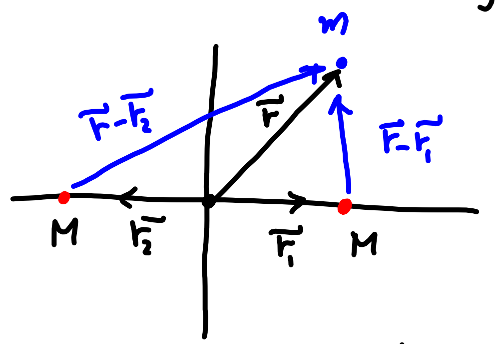
Solving for explicitly time-dependent force problems : $\vec{a}(\vec{r}, t)$

Gravitational 3-body problem ↔ no analytical solution

Sun - earth - moon

↪ most orbits are chaotic !

2) Two-center problem - two gravitational attraction centers.



$$\cancel{m}\ddot{\vec{r}} = -GM\cancel{m}\frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|^3} - GM\cancel{m}\frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|^3}$$

choose :

$|\vec{r}_1| = |\vec{r}_2| = \frac{1}{2}$

$\vec{r}_1 = (\frac{1}{2}, 0)$

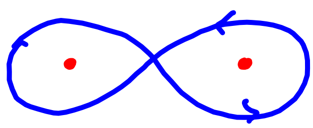
$\vec{r}_2 = (-\frac{1}{2}, 0)$

$$\boxed{\ddot{\vec{r}} = -\frac{1}{2}\left(\frac{\vec{d}_1}{|d_1|^3} + \frac{\vec{d}_2}{|d_2|^3}\right)}$$

where $\vec{d}_1 = \vec{r} - \vec{r}_1$, $\vec{d}_2 = \vec{r} - \vec{r}_2$

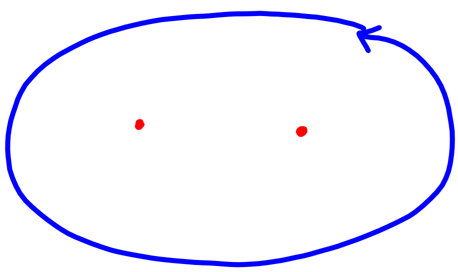
Possible orbits 1)

3)

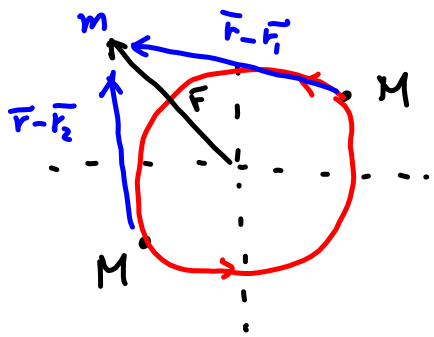


new Figure 8-orbit

2)



The circular-restricted 3-body problem



$m \ll M$

Assume that m has no effect on the motions of Ms.

$$\vec{r}_1(t) = \frac{1}{2}(\cos(\omega t), \sin(\omega t))$$

$$\vec{r}_2(t) = -\vec{r}_1(t)$$

$$\ddot{\vec{r}} = \vec{a}(\vec{r}, t)$$

$$= -\frac{1}{2}\left(\frac{\vec{d}_1(t)}{|d_1(t)|^3} + \frac{\vec{d}_2(t)}{|d_2(t)|^3}\right) \qquad \omega = 1$$

where $d_1(t) = \vec{r} - \vec{r}_1(t)$, $d_2 = \vec{r} - \vec{r}_2(t)$

$\Rightarrow$ motion of m in a time-varying gravitational field.

Symplectic algorithms for solving
a time-dependent $\vec{a}(\vec{r}, t)$

For the case Sym 1a + 1b algorithm

1a)

Cromer

$t = 0$

1) $\vec{v}_1 = \vec{v}_0 + \vec{a}(\vec{r}_0, t)\,\Delta t$ ← evaluate $\vec{a}$ at the initial time

2) $\vec{r}_1 = \vec{r}_0 + \vec{v}_1\,\Delta t$

3) $t = t + \Delta t$

for-loop

then repeat

1b)

$t = 0$

1) $\vec{r}_1 = \vec{r}_0 + \vec{v}_0\,\Delta t$

2) $t = t + \Delta t$

3) $\vec{v}_1 = \vec{v}_0 + \vec{a}(\vec{r}_1, t)$ ← evaluate at the end-time.

for-loop

Second-order $\underline{2b}$ (don't use 2a)

$\uparrow$ position - Verlet $\uparrow$ velocity - Verlet

2b) $$T_{2b} = e^{\frac{1}{2}\Delta t T} e^{\Delta t V} e^{\frac{1}{2}\Delta t T}$$

$t = 0$

Suzuki's Rule: update t after every updating of $\vec{r}$, with the coefficient of v.

$$\vec{r}_1 = \vec{r}_0 + \frac{1}{2}\Delta t \vec{v}_0$$

$t = t + \frac{1}{2}\Delta t$

$$\vec{v}_1 = \vec{V}_0 + \Delta t \, \vec{a}(\vec{r}_1, t)$$

$$\vec{r}_2 = \vec{r}_1 + \frac{1}{2}\Delta t \, \vec{v}_1$$

$t = t + \frac{1}{2}\Delta t$

for-loop

Algorithm 2a : Velocity-Verlet

$$T_{2a} = e^{\frac{1}{2}\Delta t V} e^{\Delta t T} e^{\frac{1}{2}\Delta t V}$$

$t=0$

Algorithm

$$\vec{v}_1 = \vec{v}_0 + \frac{1}{2}\Delta t\, \vec{a}(\vec{r}_0, t)$$

$$\vec{r}_1 = \vec{r}_0 + \Delta t\, \vec{v}_1$$

$t = t + \Delta t$

$$\vec{v}_2 = \vec{v}_1 + \frac{1}{2}\Delta t\, \vec{a}(\vec{r}_1, t)$$

for-loop

update t after each position updating with the coefficient of v.

Solve time-dependent Hamiltonians

$$H(t) = T + V(t)$$

Recall previously $\frac{dW}{dt} = HW \longrightarrow \frac{dW}{W} = H\,dt$

$$\Rightarrow \boxed{W(t) = e^{\int_0^t dt' H(t')} W(0)}$$

$d \ln W = H\,dt$

$\ln W = \int H(t)\,dt + C$

true only for $H(t', q, p)$ a fct, no operators.

Solve $\frac{dw}{dt} = \hat{H}(t) w$ $\qquad \hat{H}$ = operator $= T + V(q)$

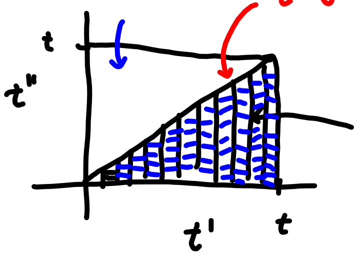
$w(t) = \mathcal{U}(t) w(0)$ $\qquad \sum_i v_i \frac{\partial}{\partial q_i}$ $\qquad \sum_i a(q_i;t)\frac{\partial}{\partial v}$

evolution operator $\quad \int_0^t dt \frac{d\mathcal{U}}{dt} \cancel{w(0)} = \int_0^t \hat{H}(t)\mathcal{U}(t)\cancel{w(0)}\, dt$

$$\mathcal{U}(t) - \underbrace{\mathcal{U}(0)}_{=1} = \int_0^t \hat{H}(t')\mathcal{U}(t')\,dt'$$

$$\mathcal{U}(t) = 1 + \int_0^t \hat{H}(t')\mathcal{U}(t')\,dt'$$

$$= 1 + \int_0^t \hat{H}(t')\left[1 + \int_0^{t'} \hat{H}(t'')\mathcal{U}(t'')\,dt''\right]dt'$$



$$= 1 + \int_0^t \hat{H}(t')\,dt' + \int_0^t dt' \int_0^{t'} dt''\, \hat{H}(t')\hat{H}(t'') + \cdots \quad t'>t''$$

$$= \int_0^t dt'' \int_0^{t''} dt'\, \hat{H}(t'')\hat{H}(t') \quad t'<t''$$

$$= 1 + \frac{1}{2}\int_0^t dt'' \int_0^t dt'\, \hat{H}(t'')\hat{H}(t')$$

$$= 1 + \frac{1}{2}\left[\int_0^t dt'\, \hat{H}(t')\right]^2 + \cdots$$

$$= e^{\int_0^t dt' \hat{H}(t')}$$

$$= 1 + \frac{1}{2}\int_0^t dt'' \int_0^t dt'\; T[\hat{H}(t')\hat{H}(t'')]$$

Define: time-ordering of operators: If $\hat{H}(t')\hat{H}(t'') = \hat{H}(t'')\hat{H}(t')$, i.e. $[\hat{H}(t'), \hat{H}(t'')] = 0$

$$T[\hat{H}(t')\hat{H}(t'')] = \begin{cases} \hat{H}(t')\hat{H}(t'') & \text{when } t'>t'' \\ \hat{H}(t'')\hat{H}(t') & \text{when } t'<t'' \end{cases}$$

if true

but $[T + V(t'), T + V(t'')] \neq 0$

$$\left[v\frac{\partial}{\partial q} + a(t')\frac{\partial}{\partial v}\right]\left[v\frac{\partial}{\partial q} + a(t'')\frac{\partial}{\partial v}\right]$$

$$v\frac{\partial}{\partial q} v\frac{\partial}{\partial q} \qquad a(t')\frac{\partial}{\partial v}\, a(t'')\frac{\partial}{\partial v}$$

With time-ordering

$$\mathcal{U}(t) = 1 + \int_0^t \hat{H}(t')dt' + \frac{1}{2}\int_0^t dt' \int_0^t dt'' \, T\left[\hat{H}(t')\hat{H}(t'')\right] + \frac{1}{3!}\int_0^t dt_1 \int_0^t dt_2 \int_0^t dt_3 \, T\left[\hat{H}(t_1)\hat{H}(t_2)\hat{H}(t_3)\right] + \cdots$$

$$\mathcal{U}(t) = T\left(e^{\int_0^t dt' \hat{H}(t')} \right)$$

← Seems difficult to implement

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↑ Japanese

$$= T\left(e^{\Delta t \sum_{k=0}^{n} \hat{H}(k\Delta t)} \right)$$

$$= e^{\Delta t \hat{H}(t)} e^{\Delta t \hat{H}(t-\Delta t)} \cdots e^{\Delta t \hat{H}(0)}$$

↑ manifestly time-ordered!