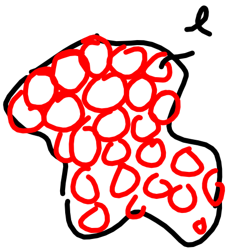
The end states of chaotic maps are self-similar fractal !

Fractal (non-integral) dimensions :

Given an object, to find its dimension,

cover it by a d-dimension "sphere" of radius ℓ



$N(\ell)$ = # of ↗ that can cover the object.

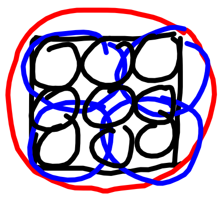
$$\lim_{\ell \to 0} N(\ell) = \text{const} \left(\frac{1}{\ell}\right)^D$$ ← dimension

$$\ln N = D \ln\left(\frac{1}{\ell}\right)$$

$$\boxed{D = \lim_{\ell \to 0} \frac{\ln N}{\ln\left(\frac{1}{\ell}\right)}}$$

Ex 1: unit sq.

$r = \sqrt{2}$



ℓ	N
1	1
1/2	4
1/3	9
⋮ $\frac{1}{n}$	⋮ n^2

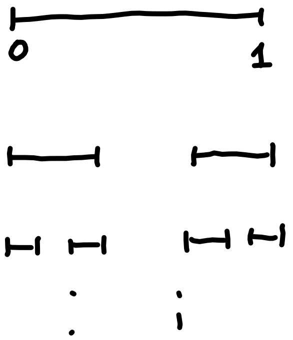
$\ell \to 0$

$n \to \infty$

$$D = \lim_{\ell \to 0} \frac{\ln(n^2)}{\ln(n)} = \frac{2 \ln n}{\ln n} = 2$$

Ex. 2 Cantor Set (throw out the middle $\frac{1}{3}$)

[0,1]

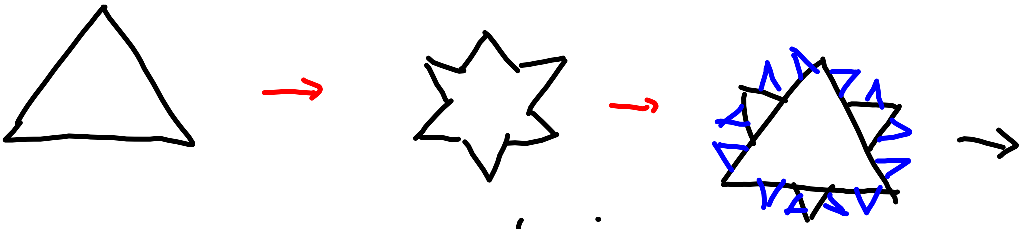


ℓ	N
1	1
$\frac{1}{3}$	2
$\frac{1}{9}$	4
$\vdots$	
$\frac{1}{3^n}$	2^n

$$\lim_{n\to\infty} D = \frac{\ln 2^n}{\ln 3^n} = \frac{\cancel{n}\ln 2}{\cancel{n}\ln 3} = 0.631\ldots$$

Ex. 2 Koch curve → coastline ← fractal dimension → $D > 1$

↳ add 1/3 → 4 sided



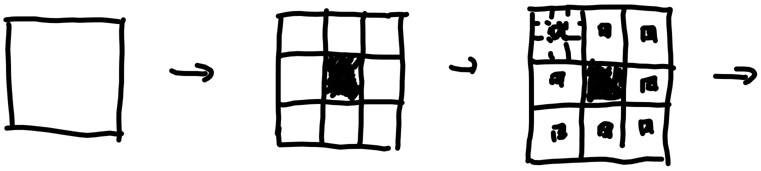
continous, but not differentiable.

ℓ	1	$\frac{1}{3}$	$\frac{1}{3^2}$	..	$\frac{1}{3^n}$
$N(\ell)$	3	$3 \cdot 4$	$3 \cdot 4^2$		$3 \cdot 4^n$

$$D = \lim_{\ell \to 0} \frac{\ln N}{\ln(1/\ell)} = \lim_{n \to \infty} \frac{\ln 3 + n \ln 4}{n \ln 3} = \frac{\ln 4}{\ln 3}$$

$$= 1.2618\ldots\ !$$

Ex. 3 Sierpinski carpet $\leftrightarrow$ 2 D Cantor set

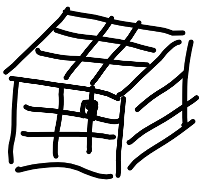


$\to$ etc.

ℓ	1	$\frac{1}{3}$	$\frac{1}{3^2}$	$\cdots$	$\frac{1}{3^n}$
N	1	8	8^2		8^n

$$D = \lim_{n\to\infty} \frac{n \ln 8}{n \ln 3} = \frac{\ln 8}{\ln 3} = 1.89\ldots$$

generalize to 3 D



throw out the middle

$$D = \frac{\ln 20}{\ln 3} = 2.72\ldots$$

All these fractal dimension objects are self-similar at all scales.

Idea → produce fractals as the end states of a dynamics

↑ simplest possible "dynamics"

↳ rule for moving to the next time step

The simplest dynamics

↪ fractal end state

↪ fixed pt.

Just consider: Linear (affine) maps

$$x_{n+1} = a x_n + b$$

with fixed pts

$$x^* = a x^* + b$$

$$\Rightarrow \quad x^* = \frac{b}{1-a}$$

only stable if $|a| < 1$

To see how the fixed pt x^* is approached:

$|f'(x^*)| < 1$

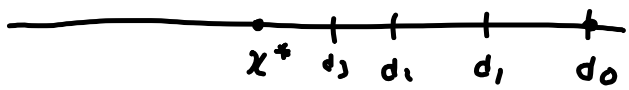
$$x_n = x^* + d_n$$

← distance from x^*

$$\cancel{x^*} + d_{n+1} = a(\cancel{x^*} + d_n) + \cancel{b}$$

$$d_{n+1} = a d_n = a^{n+1} d_0$$

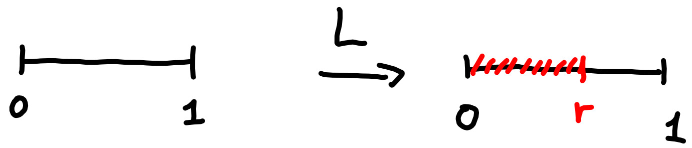
$|a| < 1$, attractive stable fixed pt.



$|a| < 1$

Ex. 1 $\quad L: \quad x_{n+1} = r\, x_n \qquad r < 1$

$\hookrightarrow$ with $x^* = 0$



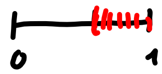
Ex. 2 $\quad R: \quad x_{n+1} = (1-s)\, x_n + s$

with $x^* = \frac{b}{1-a} = \frac{s}{s} = 1$

Ex. 3 Define $L+R$ as apply L or R with equal probability. What is the fixed pt ?

Let's take $r = \frac{1}{3}$, $s = \frac{2}{3}$

 $L \to$ 

 $R \to$ 

Starting at 

$(L+R)$ either L or R 

$(L+R)^2 = LL + LR + RL + RR$ 

LL RL LR RR

The end state of this map is the Cantor set, a fractal!