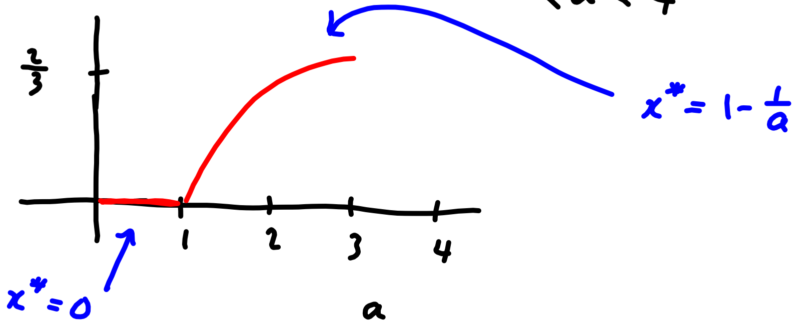
The fixed pts of the logistic map

$$x_{n+1} = a\, x_n (1-x_n) \quad \Rightarrow \quad x^* = a\, x^* (1-x^*)$$

$$1 < a < 4$$

bifurcation diagram



What is (are) the fixed pts,

or stationary population beyond $a = 3$?

Recall that the stability of $x^* = 1 - \frac{1}{a}$ is given

by $\quad f'(x^*) = 2 - a$

For $a>3$, population will oscillate between 2 fixed pts x_1^*, x_2^*

$$x_2^* = f(x_1^*) = a x_1^*(1-x_1^*)$$
$$x_1^* = f(x_2^*) = a x_2^*(1-x_2^*)$$

divide out the single fixed pt solution $x_2^* = x_1^*$

$$x_2^* - x_1^* = -a(x_2^* - x_1) + a(x_2^{*2} - x_1^{*2})$$

$$1 = -a + a(x_2^* + x_1^*) \Rightarrow$$ Sum

$$x_1^* + x_2^* = \frac{1+a}{a}$$

product:

$$x_1^* x_2^* = a^2 x_1^* x_2^* (1-x_1^*)(1-x_2^*)$$
$$1 = a^2 \left(1 - (x_1^* + x_2^*) + x_1^* x_2^*\right)$$
$$= a^2\left(1 - \frac{1+a}{a} + x_1^* x_2^*\right)$$
$$= a^2\left(-\frac{1}{a} + x_1^* x_2^*\right)$$

$$x_1^* x_2^* = \frac{1}{a^2} + \frac{1}{a} = \frac{1+a}{a^2}$$

Recall

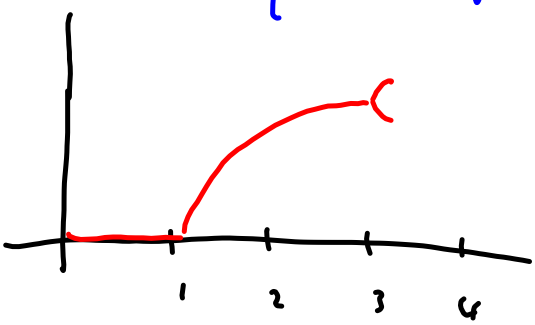
$$x^2 - Sx + P = 0$$
$$(x-x_1)(x-x_2) = 0$$
$$x = \frac{S \pm \sqrt{S^2 - 4P}}{2}$$

$$\Rightarrow x_1^*, x_2^* = \frac{1}{2}\left[\frac{1+a}{a} \pm \sqrt{\left(\frac{1+a}{a}\right)^2 - 4\,\frac{1+a}{a^2}}\right]$$

$$x_1^*, x_2^* = \frac{1}{2a}\left[1+a \pm \sqrt{(1+a)^2 - 4(1+a)}\right]$$

$$= \frac{1}{2a}\left[1+a \pm \sqrt{(1+a)(a-3)}\right]$$

exist only for $a>3$



However, the stability of x_1^*, x_2^* is given by

$$x_1^* = f(x_2^*) = f(f(x_1^*)) = F(x_1^*)$$

The stability is given $F'(x_1^*) = f'(x_2^*) f'(x_1^*)$

$$-1 < F' < 1 \qquad = a(1-2x_2^*) a(1-2x_1^*)$$

$$= a^2 (1 - 2(x_1^* + x_2^*) + 4x_1^* x_2^*)$$

$$F'(x_1^*) = 1 - (1+a)(a-3)$$

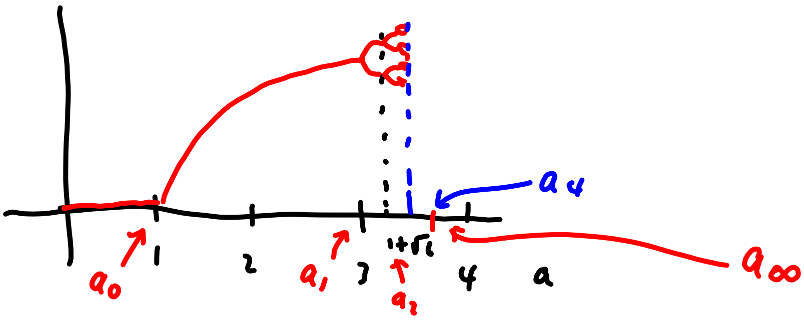
stable until $\quad 1-(1+a)(a-3) = -1$

$$1 - (-3 - 2a + a^2) = -1$$

$\Rightarrow \quad a^2 - 2a - 5 = 0 \quad 1 + 3 + 2a - a^2 = -1$

$$a = \frac{2 \pm \sqrt{4 + 4 \cdot 5}}{2} = 1 \pm \sqrt{6} = 1 + \sqrt{6} = 3.4495$$

The pattern of bifurcation



a_0 = beginning of the 2^0 stable fixed pt. $= 1 = 1$

a_1 = 2^1 " " $= 3 = 3$

a_2 = " 2^2 " " $= 1+\sqrt{6} = 3.4495$

$a_3 = 3.5441$

$a_4 = 3.56441$

$a_5 = 3.56875\ldots$

$\vdots$

infinite bifurcations at $a_\infty = 3.5699\ldots$

The pattern, or universality of the bifurcation diagram:

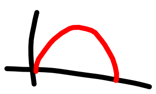
define $c_0 = a_1 - a_0 = 3 - 1 = 2$

$c_1 = a_2 - a_1 = 0.4495$ $\qquad \frac{c_0}{c_1} = 4.449$

$c_2 = a_3 - a_2 = 0.0946$ $\qquad \frac{c_1}{c_2} = 4.7516$

$c_3 = a_4 - a_3 = 0.0203$ $\qquad \frac{c_2}{c_3} = 4.6578$

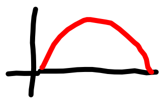
$f(x) = ax(1-x)$



$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} \to \delta = 4.6692$$

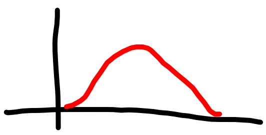
↑ Feigenbaum's constant

but take $f(x) = a\sin(\pi x)$



→ gives exactly the same

for any



Universality

bifurcation route to chaos.

The <u>superstable fixed pts</u>

$$f(x,a) = ax(1-x)$$

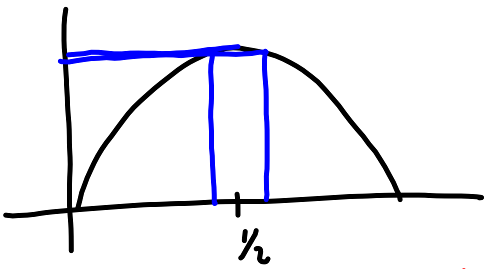
$$f'(x,a) = a(1-2x)$$

at whatever a

$x^* = \frac{1}{2}$

$f'(x^*, a) = 0$!

fixed pts near $x = \frac{1}{2}$ will condense



$\epsilon f'$

$+ \frac{1}{2}\epsilon^2 f''$

↪ shading, bright bands are due to iteration of $x^* = \frac{1}{2}$

$$f(a, \tfrac{1}{2}) = a \underset{x}{\tfrac{1}{2}} (1 - \tfrac{1}{2}) = \frac{a}{4}$$